

PROGRAM

Classical and Quantum Dynamics of Complex Systems and Applications

March 22nd - April 1st, 2021

Tuesday, March 23rd

10:00 Hirdesh Pharasi

Title: Correlation matrix analysis of complex systems

Abstract: The understanding of complex systems has become a central issue due to their existence in a wide range of scientific disciplines. Correlations among constituents have been found to be a very powerful tool for the analysis of these systems. The properties of correlation structures and their evolution with time are useful for extracting important information of the underlying system dynamics. The correlation structures change drastically during critical events which have far-reaching socio-economic effects. Thus, it is important to detect their signatures for timely intervention or prevention. In this talk, I will present some of our related studies of complex systems, with financial market as a paradigm.

11:30 Raul Hernandez

Title: A simple data analysis of daily uninterrupted trends durations of financial indices.

Abstract: In financial time series there are time periods in which market indices values or assets prices increase or decrease monotonically. We call those events “price runs” or “uninterrupted trends”. In the present paper we study the distribution of the duration of “uninterrupted” trends for the daily indices DJIA, NASDAQ, IPC and Nikkei 225 along the period of time from 30/10/1978 to 08/07/2020 and we use the simple geometric statistical model consistent with the EMH to quantify trend duration and direction of financial time series, and compare this theoretical model with the empirical data.

13:00 Suchetana Sadhukhan

Title: Dynamics of financial markets in correlation matrix space

Abstract: With a natural urge to understand the dynamics of complex systems, the analysis of correlations between constituents is crucial. Based on the similarity of the correlation matrices, markets can be characterized into “states” or “regions” through which it evolves over time. My talk will emphasize the understanding of the system based on the occurrences of correlation structures as well as the dynamical evolution of trajectories in the space of states. It is important to note here that the dynamics are in the full space; that one can visualize by dimensional scaling and the large distances are well preserved. Through comparison of the market at different time horizons in the correlation matrix space, this study provides a better understanding of the system.

Thursday, March 25th

10:00 David Davalos, IFUNAM

Title: Quantum channels: properties and applications

Abstract: Quantum operations, also called quantum channels, describe the most general linear operations that quantum systems can undergo, and can be seen as the building blocks of quantum dynamical processes. In this talk I will give a small overview of quantum channels and discuss some of their remarkable properties. Particularly I will show some results regarding divisibility types and how they unveil non-convex structures inside the convex set of quantum channels, and how these types are related to Markovianity. Furthermore, as quantum operations capture the mathematical aspects of irreversibility and information loss in the quantum realm, I show that they can be applied to describe imperfect quantum tomography and coarse-graining. Additionally, I discuss the symmetries and invariant spaces of channels describing particle indexing-errors during quantum tomography.

11:30 Antonio Rosado, IFUNAM

Title: Density operator formalism for Dirac particles: Dynamics and the connection with open quantum systems in the non-relativistic limit

Abstract: In this work we introduce the correct density operator formalism used to describe relativistic particles of spin $1/2$, we shall present the general form of the density operator for mixed states as well as its covariant transformation laws. We shall postulate its dynamical equation as a generalization of the standard Dirac equation and analyze the allowed form a possible source term must have in order to preserve relativistic covariance. Such analysis will lead us to show the incompatibility between the postulates of special relativity and of open quantum systems. From the viewpoint of classical field theory we shall discuss the inexorable closedness of a relativistic system and parting from this fact we will show how to recover the typical Lindblad equation taking the non relativistic limit in the equation of motion of the **covariant density operator**.

12:30 Yael Hernández, IFUNAM

Title: Hexagonal rings: an algebraic description of the electronic structure with applications to benzene and borazine

Abstract: A tight-binding model for an hexagonal ring is presented. The doublets in the spectrum are explained using concepts of group theory and recognizing the C_6 symmetry in benzene and C_3 in borazine. Symmetry breaking is explored by 1) introduction of one defect on site, 2) couplings modifications and 3) time reversal symmetry breaking i.e. the introduction of a magnetic field piercing the molecular plane. Lifted doublets are obtained as consequence. The supersymmetry in a 6-ring is used to explain the level structure of the deformed molecules. Applications to benzene and borazine energy levels are presented.

1:30 Israel Castro, IFUAP

Title: An electromagnetic Maxwell's demon.

Abstract: The formulation of a Maxwell demon is studied from a Hamiltonian point of view, involving an arbitrarily large ensemble of independent particles. Subsequently, the Schrödinger equation is used to characterise its wave dynamics with apparently decreasing entropy along the arrow of time, and thus be able to construct a system capable of cooling and ordering classical radiation.

Friday, March 26th

10:00 Alf Löffler, Germany

Title: Effects of gaseous emissions on average immissions of nitrogen-dioxide NO₂.

Abstract: Objective is a multivariate model describing the dependence of average of immissions on

- emissions
 - traffic
 - heating
 - industry
 - natural sources
- atmospheric conditions
 - wind
 - humidity
 - temperature
 - solar radiation
- other atmospheric components
 - water vapor (H₂O)
 - ozone (O₃)
 - nitrogen-oxide(NO)
 -

Immissions can be measured using chemiluminescence (excited molecules decay to ground state). Long time-series of measured values need to be cleaned from noise (filtering, smoothing, detection of patterns). Using statistical linear and general models are useful to find

- longterm trends
- seasonal periodicity
- and anomalies (non periodic events)

Correlation with characteristics of emission sources (seasonal dependence of heating, weekly cycle of traffic volume, efficiency of motorcars) do show the dependency of immissions on various factor. They may be quantitative expressed by the model formula.

11:30: Suchetana Sadhukhan, ICF-UNAM

Title: Dynamical studies of pollution near and far from streets and highways in Baden Württemberg, Germany

Abstract: I will report an ongoing work, directly related to Dr. Alf Löffler's talk, using the same hourly air pollution data from Baden-Württemberg, Germany. I will emphasize intraday

patterns of the pollutants' concentration (contraction and expansion period) which are influenced by daily temperature variability, illumination changes, humidity, human interventions mainly come from vehicles and industrial pollution, etc. A systematic understanding of the relationship between air pollutant concentration and meteorological conditions is the prerequisite and basis for scientifically formulating air pollution prevention and control policies, hence deserves further research.

12:00: Parisa Majaari

Title: Sectors behavior of the financial markets:

Abstract: In recent decades the stock market has begun and is gaining a particularly important role for economy. Examining this market is one of the most common issues which concerns analysts especially after the opening of markets where the interactions of the world's economies are much more intense. We study the sectors behavior of S&P 500 (USA) and Nikkei 225 (JPN) markets. Averaging over the correlation matrix for each block which belongs to different sectors provided a useful tool for studying sectoral behavior. Through the analysis, we study the correlations of the sectoral index traded in the 14-year period and we use clustering analysis by help of k-means clustering.

13:00 Emerson Sadurní, IFUAP

Title: SIR models: Pandemic forecasting and its limitations

Abstract: In this talk I will present a class of exactly solvable epidemic models and their extensions. These include systems with three species (the typical SIR model), systems with time-dependent infection rates (quarantine policies) and systems with interacting regions (migration). In the context of the COVID19 pandemic, the case of Mexico will be treated up to november 2020. Then I will indicate some shortcomings of the models in the description of explosive events such as black friday ("buen fin" in Mexico) and seasonal festivities. I will conclude with an apparent conflict between limited spread velocities and imperfect data acquisition.

Monday, March 29th

10:00 David Dávalos

Title: Quantum channels: properties and applications

Abstract: Quantum operations, also called quantum channels, describe the most general linear operations that quantum systems can undergo, and can be seen as the building blocks of quantum dynamical processes. In this talk I will give a small overview of quantum channels and discuss some of their remarkable properties. Particularly I will show some results regarding divisibility types and how they unveil non-convex structures inside the convex set of quantum channels, and how these types are related to Markovianity. Furthermore, as quantum operations capture the mathematical aspects of irreversibility and information loss in the quantum realm, I show that they can be applied to describe imperfect quantum tomography and coarse-graining. Additionally, I discuss the symmetries and invariant spaces of channels describing particle indexing-errors during quantum tomography.

11:30 Enrico Scalas

Title: The Mathematics of Human Contact.

Abstract: After discussing some recent theoretical developments concerning the mathematics of human contact, I provide a statistical analysis of high-resolution contact pattern data within primary and secondary schools as collected by the SocioPatterns collaboration. Students are graphically represented as nodes in a temporally evolving network, in which links represent proximity or interaction between students. I focus on link- and node-level statistics, such as the on- and off-durations of links as well as the activity potential of nodes and links. Parametric models are fitted to the on- and off-durations of links, inter-event times and node activity potentials and, based on these, I propose a number of theoretical models that are able to reproduce the collected data within varying levels of accuracy. By doing so, I aim to identify the minimal network-level properties that are needed to closely match the real-world data, with the aim of combining this contact pattern model with epidemic models in future work.

This is joint work with Stephen Ashton, Istvan Z. Kiss and Nicos Georgiou

13:00 Hirdesh Pharasi

Title: A random matrix approach to Market states

Abstract: Financial markets show a number of non-stationarities, ranging from volatility fluctuations over ever-changing technical and regulatory market conditions to seasonalities. On the other hand, financial markets show various stylized facts which are remarkably stable. It is thus an interesting question to find out how these stylized facts, a simplified presentation of an empirical finding, emerge. Here in the talk, I will discuss preliminary investigations of stylized fact corresponding to market states which shows qualitatively similar clustering obtained from the empirical data of two financial markets of S&P 500 (USA) and Nikkei 225 (JPN) markets over a period of 2006-2019. We investigate the statistical properties of correlation matrices constructed from the sliding epochs and classified the similar correlation matrices into groups named *market states*. We analyze the same clustering technique on surrogate data and the fluctuations arise due to the white noise of short time series. We use the correlated Wishart orthogonal ensemble for the construction of surrogate data whose average correlation equals the average of the real data.

Tuesday, March 30th

10:00 Zeidy Muñoz Torres, Facultad de Psicología y Centro de Ciencias de la Complejidad (C3), UNAM

Title: Organization of Neuronal Activity in Men and Women with Sleep Apnea

Abstract: Sleep apnea (SA) is characterized by repetitive cessations (apnea) or reductions (hypopnea) of breathing during sleep with higher prevalence in men than in women. Sex steroidal hormones have been involved on breathing regulation thus, the underlying pathophysiology may vary considerably between the sexes. We compared the electrical brain activity by means of electroencephalogram (EEG) across the night and during wakefulness between men and women with moderate and severe sleep apnea. Our study

emphasizes the importance of understanding the differential effects of sleep disorders on men and women to develop more precise diagnostic criteria according to gender, including quantitative EEG analysis tools.

This work was financed partially by Universidad Nacional Autónoma de México, PAPIIT: IA208018.

11:00 Victor Hugo Martin del campo Moreno, Universidad de Guadalajara

Title: Energy minimizing dynamics of binary spin systems on complex networks

Abstract: In the past, zero-temperature quenching dynamics of the Ising model on complex networks have been applied to the field of opinion dynamics to model the spreading of opinions. We will introduce a new dynamical rule inspired from how an individual is more disposed to influence other agents in his social system when he/she is supported. We compare the final structure of the system using the zero-temperature Glauber dynamics and the proposed dynamical rule in two types of small-world networks, known as Watts-Strogatz network and Euclidean-type network. We will present the changes in the final structure due to higher values of the average degree.

11:30 Carlos Manuel Rodríguez Martínez, UV

Title: Search of stock trading strategies by Grammar-based Genetic Programming

Abstract: In this work, we make use of Grammar-based Genetic Programming to construct intraday trading strategies using a wide array of technical indicators combined by boolean operators, where the formal grammar acts as a blueprint of the type of strategies that traders usually construct in their operations. The goal of the paper is twofold. Firstly, to analyze the applicability and profitability of trading strategies based on price and volume-related technical indicators. Secondly, to show how a complete, autonomous and self-adapting trading system can be constructed to operate on a stock market.

12:30 Laura Pérez Arvisu, Universidad de Guadalajara

Title: Trajectories of Cellular Automata in an Environment Governed by the Rules of Conway's Game of Life.

Abstract: In this project we study the behavior of cellular automata in the cellular environment governed by John Conway's rules of the game of life. We are developing this through 3 points: We have study the effects that periodic boundary conditions and other board disturbances produce on the density of life in the map. Subsequently, we write a program for a deterministically self-directed cellular automaton in the cellular environment governed by the rules of John Conway's game of life. Finally, a human-driven version was programmed for the collection of information and study of the trajectories of the manually steered agent.

13:00 Horacio Tapia-McClung, UV

Title: Empirical Spectral Distributions of Symmetric Sparse Matrices from Cellular Automaton

Abstract: Having observed that some statistical features of time series derived from a Cellular Automaton, the Game of Life, reproduce empirical facts observed in real financial time series, we look at it by means of the empirical spectral distributions of some symmetric sparse matrices obtained from the spatial configurations that result from the evolution of the system. Using numerical simulations based on the inversion formula of the sample Stieltjes transform it is possible to explore the requirements that lead to the recovery of the known theoretical distributions. When we use this approach to the CA case(s) we can see good agreement with the numerical empirical results despite not having a clue of what the limiting distribution of eigenvalues is.

Wednesday, March 31st

10:00 Thomas Gorin (part I); Alejandro Contreras (part II)

Title: Normally distributed random unitaries in $SO(3)$

Abstract: We introduce "normal distributions" for general unitary transformations in $SO(3)$ as a means to describe errors in the processing of qubits. We show that the Peter-Weyl theorem allows to calculate analytically the parameters of the corresponding quantum channel as well as many related quantities. If centered on the identity, random normally distributed transformations are a subset of the Pauli channels. However, in the case of non-centered distributions (implying a drift) this is no longer the case. We then turn to situations, where random unitaries are applied to different qubits, as they are used in error-correcting or entanglement distillating protocols. Here, the normal distributions allow to implement correlated errors in a natural way, similar to the standard case of real normally distributed multivariate random variables.

11:15 Parisa Maajari, ICF-UNAM

Title: Asymmetric Floquet sideband tunneling in uniaxially strained graphene

Abstract: The interplay of strain engineering and photon-assisted tunneling of electrons in graphene is considered for giving rise to atypical transport phenomena. The combination of uniaxial strain and a time-periodic potential barrier helps to control the particle transmission for a wide range of tunable parameters. With the use of the tight-binding approach, the elasticity theory, and the Floquet scattering, we found an angular shift of the maximum transmission in the sidebands for uniaxial strains breaking the mirror symmetry with respect to the normal incidence, which is called anomalous Floquet tunneling. We show that electron tunneling depends strongly on the barrier width, incident angle, uniaxial strain, and the tuning of the time-periodic potential parameters. An adequate modulation of the barrier width and oscillation amplitude serves to select the transmission in the sidebands. These findings can be useful for controlling the electron current through the photon-assisted tunneling being used in multiple nanotechnological applications. We use the benefit of the combination of uniaxial strain and a time-harmonic potential to have further control on particle transmission.

12:25 Emmanuel Paredes Rocha, ICF-UNAM

Title: Gradient-index electron optics in graphene p-n junctions

Abstract: We investigate the electron transport in smooth graphene p-n junctions, generated by gradually varying electrostatic potentials. The numerically calculated coherent current flow patterns can be understood largely in terms of semiclassical trajectories, equivalent to the ones obtained for light beams in a medium with a gradually changing refractive index. In smooth junctions, energetically forbidden regions emerge, which increase reflections and can generate pronounced interference patterns, for example, whispering gallery modes. The investigated devices do not only demonstrate the feasibility of the gradient-index electron optics in graphene p-n junctions, such as Luneburg and Maxwell lenses, but may have also technological applications, for example, as electron beam splitters, focusers, and waveguides. The semiclassical trajectories offer an efficient tool to estimate the current flow paths in such nanoelectronic devices.

13:00 Fabricio Toscano, Instituto de Física, Universidade Federal do Rio de Janeiro

Title: Quantum de Bruijn identity for Gaussian channels

Abstract: In realistic scenarios, information channels (classical or quantum) cannot be completely isolated from its surrounding environment. In fact, the systems that implement the channels, evolve in time with dynamics that are a combination of random and deterministic effects. The information-theoretical analysis of this behavior in time is crucial to improve the communication rates of these channels. Here we address this analysis in Gaussian channels for continuous variable systems that are one of the more important channels in information processing, both in the classical and quantum domains. Precisely, we aim to quantify how the information encoded in a continuous variable quantum system is affected when the system is going through a multimode bosonic Gaussian channel that is a dynamical semigroup and so modeled by a linear Lindblad master equation (LME). Our approach is based on a novel decomposition of the infinitesimal generator of any dynamical semigroup which admits a Lindblad form. This decomposition allows us to relate the rate of change of the von Neumann entropy with the divergence-based quantum Fisher information at any time on multimode bosonic Gaussian channels modeled by a linear LME. This is made in a resemblance to the classical de Bruijn identity recently generalized to Fokker-Planck channels. In addition, the identity found allows a new perspective on the analysis of stationary situations in multimode bosonic Gaussian channels that are dynamical semigroups.

14:30 Pablo Carlos López Vázquez, Centro universitario de los Valles, Universidad de Guadalajara

Title: Probing the dynamics of an dissipative harmonic oscillator via a single qubit

Abstract: We investigate strategies to monitor the dynamics and determine the characteristics of a dissipative harmonic oscillator coupled to a finite temperature heat bath. For this purpose we use a single qubit coupled via dephasing to the oscillator. The whole system is described by a quantum master equation, which can be solved analytically. The solution is surprisingly simple and allows to read off all relevant parameters from the behavior of the qubit coherence as a function of time. Analyzing the qubit coherence for a short time, turns out to be sufficient to predict the complete dynamics of the oscillator, including its asymptotic state at thermal equilibrium.

15:30 Alejandro Ramírez Yañez, Universidad de Guadalajara.

Title: Fidelity and decoherence function for the dissipative kicked harmonic oscillator (Thesis project)

Abstract: We start by considering the kicked harmonic oscillator (KHO) as a closed system with unitary dynamics. In this case, the dynamics is given in terms of a Floquet operator. We perturb the system by changing the kick strength by a small amount. Then we calculate the fidelity amplitude at multiples of the kick period.

In the second part of the work, we consider the KHO in thermal contact with a finite temperature heat bath and coupled to a two-level system. This is done in such a way (dephasing coupling) that the coherence in the two-level system becomes equal to the fidelity amplitude in the limit of the closed system (reducing the coupling to the heat bath to zero).

In all cases, we perform analytical calculations (linear response and/or time-independent perturbation theory) to compare our numerical results in a meaningful way to predictions obtained from random matrix theory.

16:00 onwards:

Discussion on all lines of research.

Thursday, April 1st

10:00 Paola Olguin

Title: Stationary pattern a new approach to study the brain dynamics

Abstract: Due to ever-changing external stimuli, noise contamination, and qualitative changes during the transition between different physiological states EEG recordings are highly non-stationary. Nevertheless, a pronounced correlation structure extremely stable in time, which is almost independent of the physiological brain state and homogeneous across subjects has been found. This stationary pattern has been interpreted as a manifestation of a dynamical ground state of the brain activity, necessary to preserve an efficient operational mode, or, expressed in terms of dynamical system theory, we interpret it as a “shadow” of the evolution on (or close to) an attractor in phase space.

Thus, non-stationary dynamical aspects of higher cerebral processes should manifest in deviations from this stable pattern. We confirm this hypothesis via a correlation analysis of EEG recordings of 10 healthy subjects during night sleep, 20 recordings of 9 epilepsy patients, and 42 recordings of 21 healthy subjects in resting state. In particular, we show that the estimation of deviations from the stationary correlation structures provides a more significant differentiation of physiological stages and more homogeneous results across subjects.

10:30 J. Daniel Arzate, Instituto de Investigación en Ciencias Básicas y Aplicadas - UAEM

Title: An EEG-fMRI study connecting large scale brain networks across time-scales

Abstract: Relating brain dynamics acting on time scales that differ by at least an order of magnitude is a nontrivial but fundamental issue in brain research. The same is true for the

observation of stable dynamical structures in otherwise highly non-stationary signals. The present study addresses both problems by the analysis of simultaneous resting-state EEG-fMRI recordings of 53 patients with epilepsy. Confirming previous findings, we observe a generic and temporally stable average correlation pattern in EEG recordings. We design a predictor for the General Linear Model describing fluctuations around the stationary EEG correlation pattern in order to detect resting state networks in fMRI data. The acquired statistical maps are contrasted to several surrogate tests and compared with maps derived by partial Independent Component Analysis of the fMRI recordings. Our results suggest that both, the stationary EEG pattern as well as resting state fMRI networks are different expressions of the same brain activity, namely the dynamics on (or close to) a stable attractor in phase space that is necessary to maintain the brain in an efficient operational mode. We show that this interpretation is congruent with the theoretical framework of complex systems as well as with the brain's energy balance.

11:30 Emmanuel Olguin

Title: Interrelationship between different frequency bands of an EEG recording - an univariant approach

Abstract: Since about a decade evidence has been accumulated that the so-called phase-amplitude coupling observed in EEG plays an important role in information processing. Here the instantaneous phase of a low frequency band modulates the instantaneous amplitude of higher frequency components.

It is believed that such a mechanism is responsible for the influence of rhythmic acoustic stimuli (which are in the low frequency range of about 1-3Hz) on brain activity operating on time scales that are about 10 times faster. Here we propose a new correlation matrix based on Pearson coefficients in order to measure possible inter-band interrelations. We further propose a well controlled numerical model in order to perform a quantitative comparison between the phase amplitude coupling and the inter-band correlations.

12:30 Alberto Isaac Aguilar

Title: Importancia de las fases de Fourier

Abstract: Al trabajar con transformada de Fourier generalmente se presta mayor atención a la amplitud dejando de lado las fases. En esta plática hablaremos sobre la importancia de las fases de Fourier y daremos ejemplos donde se muestra que gran parte de la información relevante se encuentra en las fases y no en las amplitudes. También discutiremos sobre una metodología y medidas que estamos probando para extraer la mayor información posible de las fases de Fourier aplicado al análisis de imágenes de sistemas que presentan transiciones de fase y a sistemas dinámicos.

13:30 Ana Ganriela Guerrero

Title: Preliminary Exploration of fMRI Brain Data under Various Tasks

Abstract: Functional magnetic resonance imaging, fMRI, is a technique used for both clinical and research purposes. It studies neuronal activity and allows to know the functionality of certain regions responsible for sensory, motor, cognitive and affective processes in normal and pathological brains. This work attempts to compare the statistical

distributions on connectivity, distance and eigenvalues given a threshold value in the correlation matrix in order to observe a power law for a subject under different tasks. It is observed that the result regarding the distribution of connectivity has been studied and coincides with previous works.

16:00 Diego Santiago Alarcón, Instituto de Ecología, A.C, Xalapa

Title: Quantum aspects of evolution: a contribution towards evolutionary explorations of genotype networks via quantum walks.

Abstract: Quantum biology seeks to explain biological phenomena via quantum mechanisms, such as enzyme reaction rates via tunnelling and photosynthesis energy efficiency via coherent superposition of states. However, less effort has been devoted to study the role of quantum mechanisms in biological evolution. In this paper, we used transcription factor networks with two and four different phenotypes, and used classical random walks (CRW) and quantum walks (QW) to compare network search behaviour and efficiency at finding novel phenotypes between CRW and QW. In the network with two phenotypes, at temporal scales comparable to decoherence time T_D , QW are as efficient as CRW at finding new phenotypes. In the case of the network with four phenotypes, the QW had a higher probability of mutating to a novel phenotype than the CRW, regardless of the number of mutational steps (i.e. 1, 2 or 3) away from the new phenotype. Before quantum decoherence, the QW probabilities become higher, turning the QW effectively more efficient than CRW at finding novel phenotypes under different starting conditions. Thus, our results warrant further exploration of the QW under more realistic network scenarios (i.e. larger genotype networks) in both closed and open systems (e.g. by considering Lindblad terms).

17:00 José Santiago García, Universidad Tecnológica de la Mixteca

Title: Exact crystalline propagators of some inhomogeneous crystals

Abstract: Inhomogeneous finite, infinite and semi-infinite chains with corresponding $SU(2)$, $SU(1,1)$ and Heisenberg dynamical groups are studied and the associated crystalline propagators are calculated by means of the discrete Mello-Moshinsky equations, in the case of semisimple groups, and by other standard algebraic techniques for the Heisenberg chain. Propagation properties are discussed for each chain and it is found that, when a linear potential is taken into account, Bloch oscillations take place in this type of one-dimensional lattices. The resulting dynamics contrast with that of the standard (homogeneous) tight-binding Hamiltonian. Revivals, transport of states, periodic interference patterns as well as propagation of Gaussians, are susceptible of discussions for the $SU(2)$ chain. Finally, we pave the road toward exact crystalline propagators of 2D inhomogeneous lattices with $SU(3)$ and $SO(4)$ as dynamical groups.
