Disease Transmission in Layered Networks

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Part I: Final size relations for deterministic models

The Kermak-McKendrick Epidemic Model (1927)

$$v(t) = S(t) \int_0^\infty \beta(\tau) v(t-\tau)\, d\tau$$

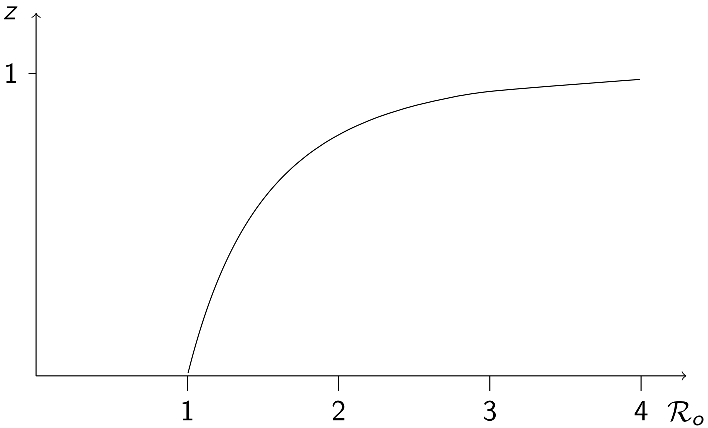
$$S'(t) = -v(t)$$

The final size, $z = 1 - \dfrac{S(\infty)}{N_0}$, is a solution of

$$\boxed{\mathcal{R}_o z + \ln(1-z) = 0}$$

where $\mathcal{R}_o = N_0 \int_0^\infty \beta(\tau)\, d\tau$

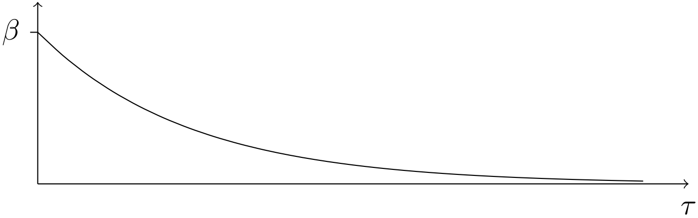
The final size relation

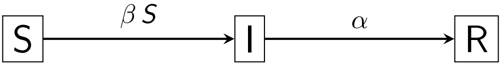


$$\mathcal{R}_o z + \ln(1 - z) = 0$$

Example 1: The SIR model

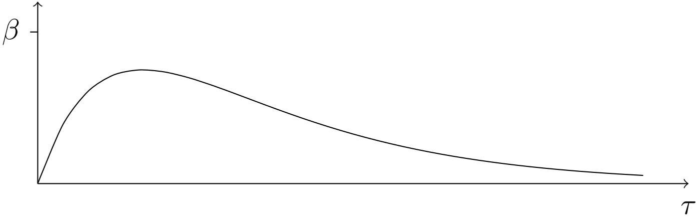
$$\beta(\tau) = \beta e^{-(\alpha+\mu)}$$

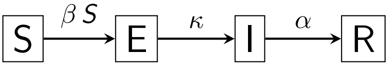




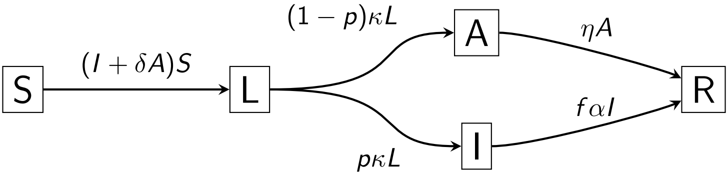
Example 2: The SEIR model

$$\beta(\tau) = \beta \frac{\kappa(e^{-\kappa\tau} - e^{-\alpha\tau})}{\alpha - \kappa}$$





Example 2: The SLIAR model



$$
\begin{aligned}
S' &= -S(I + \delta A) \\
L' &= S(I + \delta A) - \kappa L - \varphi_L L + \theta_L L_T \\
I' &= p\kappa L - \alpha I - \varphi_I I + \theta_I I_T \\
A' &= (1 - p)\kappa L - \eta A - \varphi_A A + \theta_A A_T \\
R' &= f\alpha I + f_T \alpha_T I_T + \eta A + \eta_T A_T
\end{aligned}
$$

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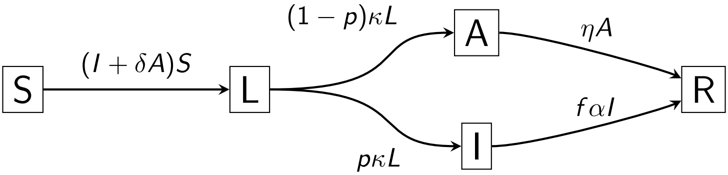

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$$S' = -S\beta^T x$$

$$x' = S\beta^T x Q - Vx$$

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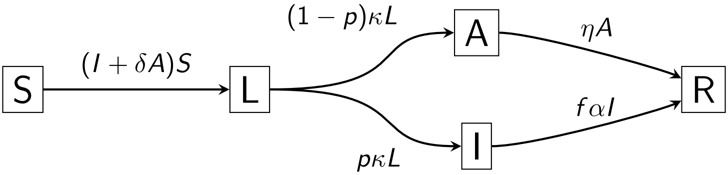

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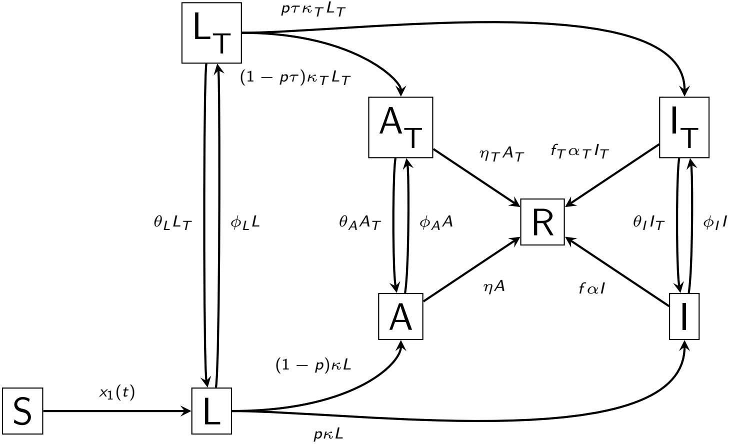
$$x' = S\beta^T x Q - Vx$$

$$\beta(\tau) = \beta^T e^{-V\tau} Q$$

Example 3: The SLIAR model with treatment



$$\beta(\tau) = (1 - \phi(\tau))\beta_U(\tau) + \phi(\tau)\beta_T(\tau)$$

The multi-type Epidemic Model

$$v_i(t) = S_i(t) \sum_{j=1}^{m} f_{ij}(N(t)) \int_0^{\infty} \beta_j(\tau) v_j(t-\tau)\, d\tau$$

$$S_i'(t) = -v_i(t)$$

The final size, $z_i = 1 - \frac{S_i(\infty)}{N_{i0}}$, is a solution of

$$\boxed{K_{ij} z_j + \ln(1 - z_i) = 0} \qquad \text{where } K_{ij} = N_{0_i} f_{ij}(N^*) \int_0^{\infty} \beta_j(\tau)\, d\tau.$$

This equation has a unique solution in $(0,1)^m$ if and only if $\mathcal{R}_o = \rho(K) > 1$.

Example 4: Age structured epidemic

Population: $N_i = (60^2 - (40 - i)^2)/266667, \;\; 0 < i < 100,$

Contact rates: $a_i = \begin{cases} 1.46 & 0 \leq i \leq 12, \\ .66 & 12 < i \leq 18, \\ .16 & 18 < i \leq 100, \end{cases}$

Infectivity: $\displaystyle\int_0^\infty \beta_j(\tau)\, d\tau = A_j = 3, \;\; 0 < j < 100.$

The final size is

$$z_i = \begin{cases} 941.50 & 0 \leq i \leq 12, \\ 722.86 & 12 < i \leq 18, \\ 267.35 & 18 < i \leq 100, \end{cases}$$

in infections per 1000 for the three age groups.

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Population: $N_i = (60^2 - (40 - i)^2)/266667, \;\; 0 < i < 100,$

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Infectivity: $\int_0^\infty \beta_j(\tau)\, d\tau = A_j = 3, \;\; 0 < j < 100.$

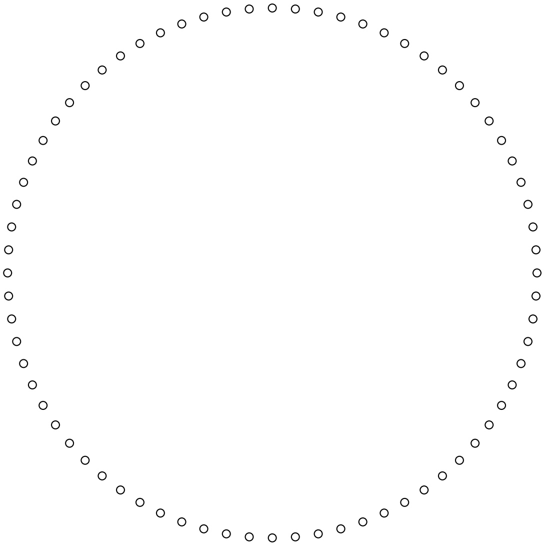
Susceptibility: $b_i = \begin{cases} 1 & i < 77, \\ 5 & i > 77 \end{cases}$

The final size is $z_i = \begin{cases} 951.35 & 0 \le i \le 12, \\ 745.03 & 12 < i \le 18, \\ 282.01 & 18 < i \le 77, \\ 863.00 & 77 < i \le 100 \end{cases}$

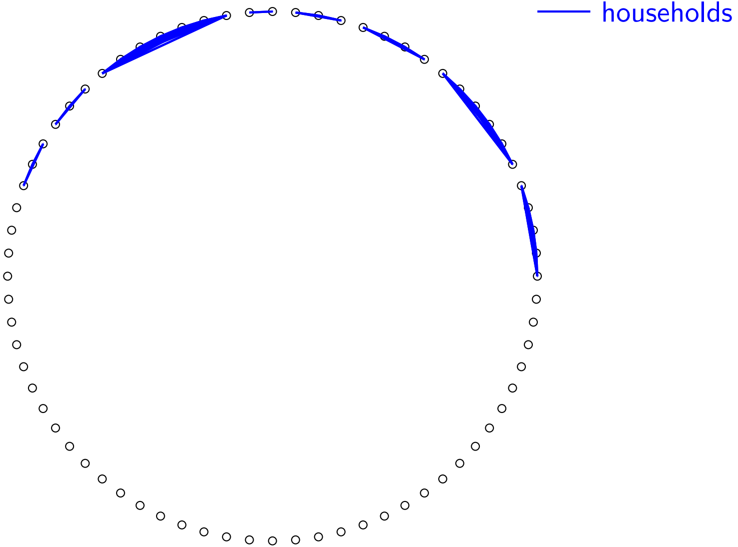
in infections per 1000 for the three age groups.

Part II: Layered Networks

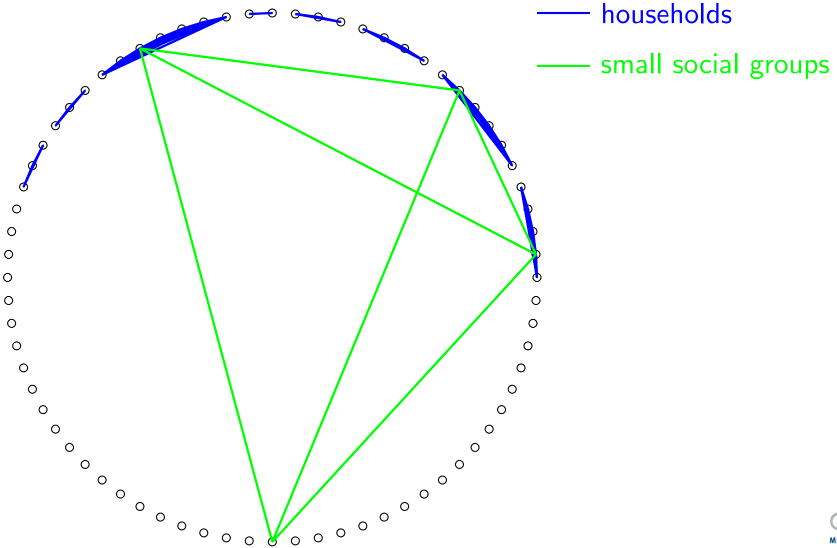
A layered network



A layered network



A layered network



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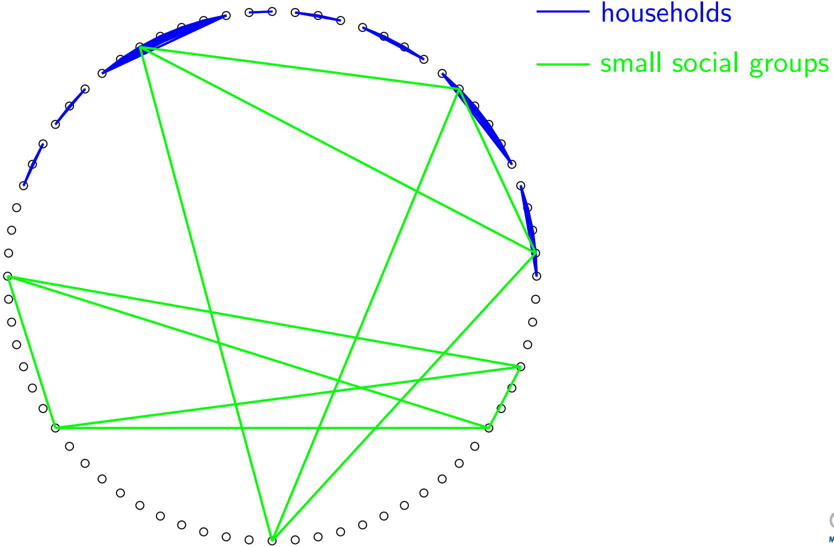
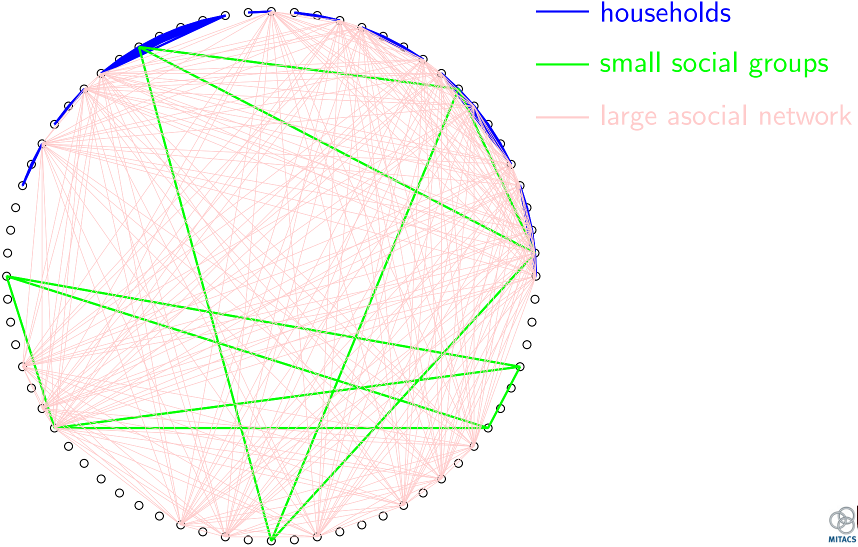
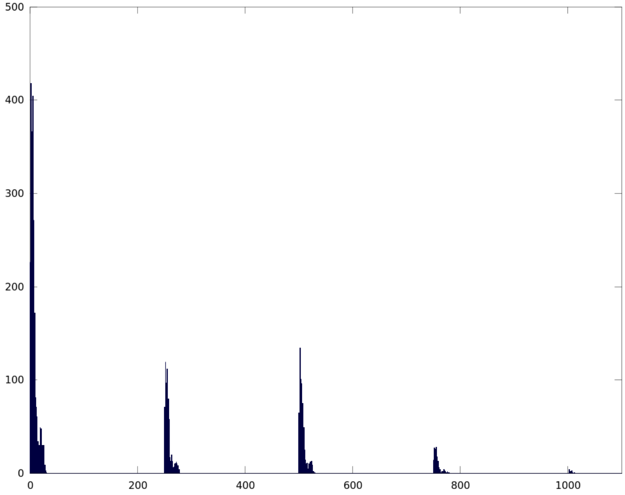


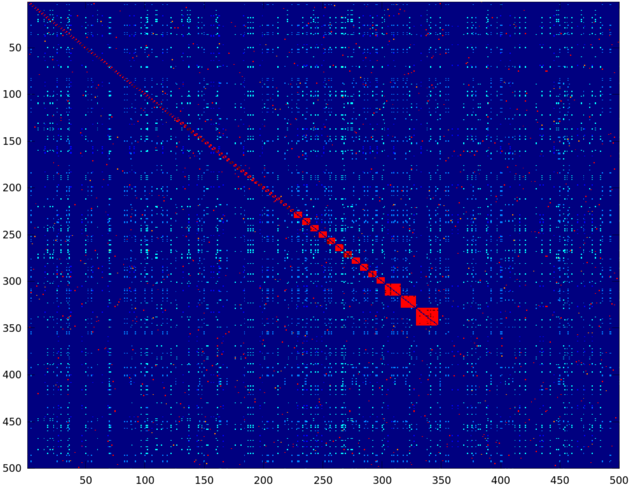
A layered network



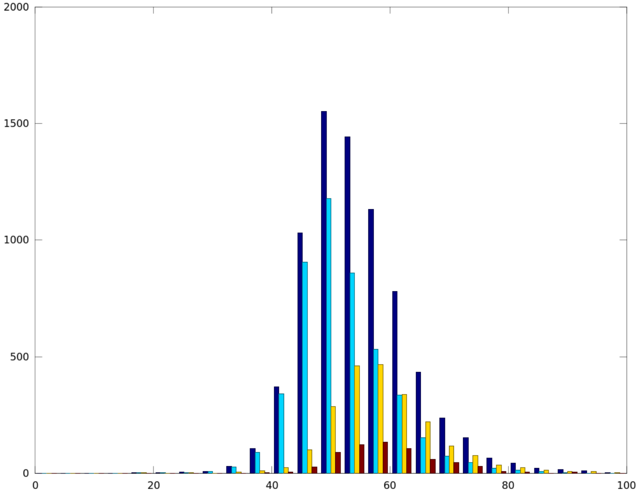
The Degree Distribution



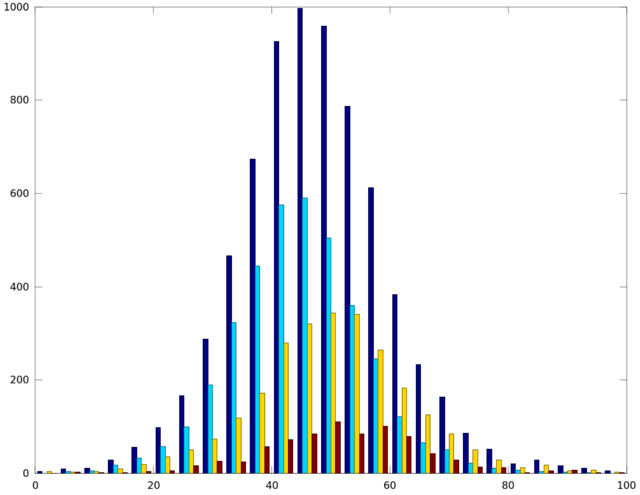
The Contact Matrix



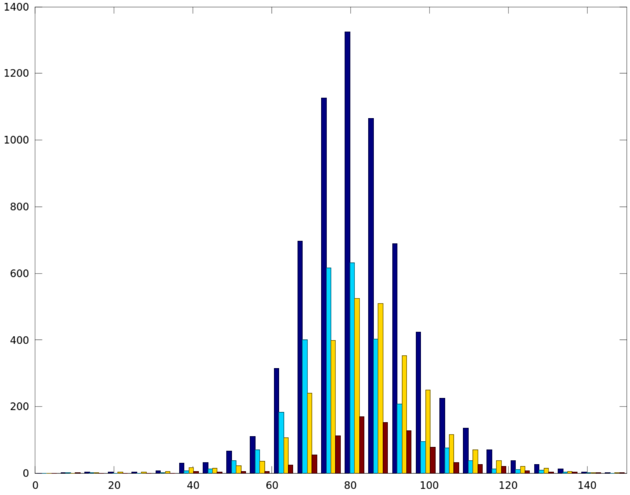
The Incidence Curves I



The Incidence Curves II



The Incidence Curves III



The Incidence Curves III