Continuous Time Evolution of Disease Spread on a Network

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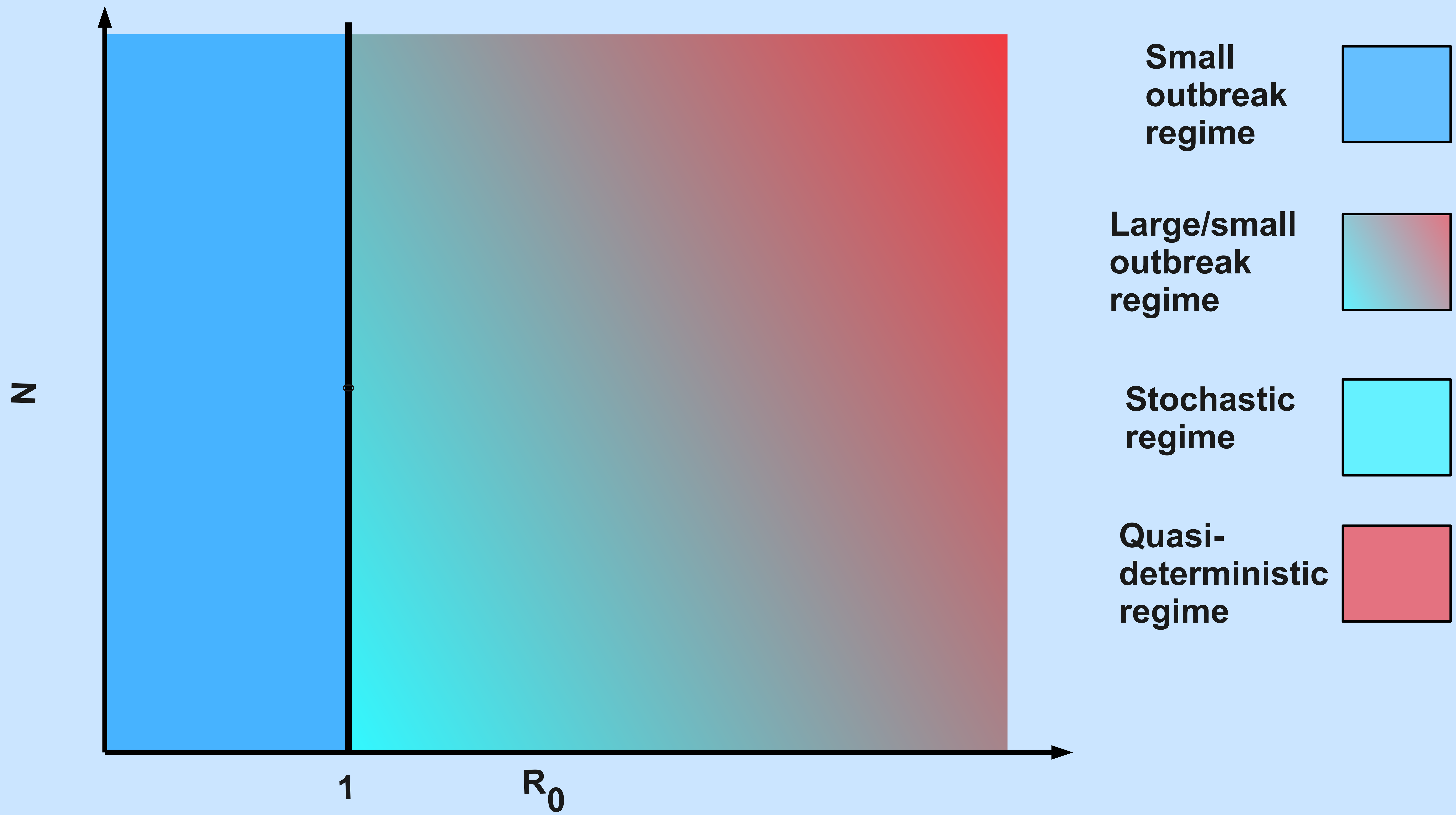
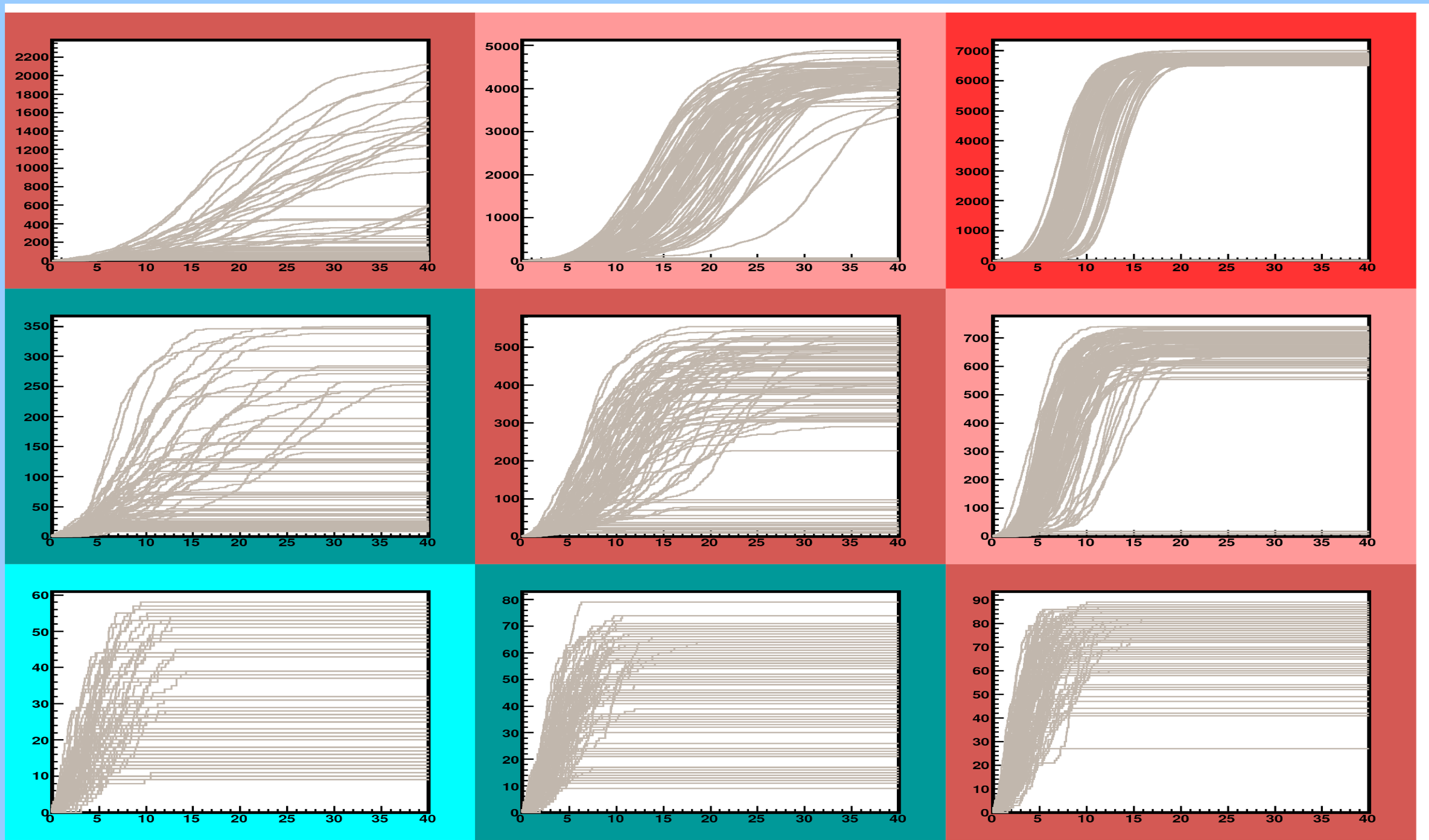
Outline

- General features of disease spread
- Network basis
- Individual disease states
- Disease transmission dynamics
- Poisson/binomial network
- General network
- Extensions
- Conclusion

General features of disease spread



General features of disease spread



Network basis

- Degree distribution p_k gives the probability that a randomly chosen vertex has degree k
- Distribution $q_k = kp_k / z$ gives the probability that a randomly chosen stub belongs to a vertex of degree k, where $z = \langle k \rangle_{p_k}$
- Excess degree $Z_x = \langle k-1 \rangle_{q_k}$ gives the average degree of a vertex, chosen by targeting one of its stub, excluding the targeted stub

Disease states

- Each individual can be susceptible (S), infected (I), or removed (R).
- $\lambda_i(\tau)$ and $\lambda_r(\tau)$ are rates for (S $\longrightarrow$ I) and (I $\longrightarrow$ R)
- $\Psi(\tau)=\exp(-\int_0^\tau \lambda_r(\tau')d\tau')$ gives the probability of being infectious up to period τ
- $\psi(\tau)=\frac{-d\Psi(\tau)}{d\tau}$ gives the probability density of being removed at time

Disease states

- Transmissibility $T(\tau) = 1 - \exp\left(-\int_0^{\tau} \lambda_i(\tau')\, d\tau'\right)$ gives the probability that disease is transmitted up to time τ
- $T(\infty)$ gives the probability of disease transmission after infectious individual is recovered
- $T = \int_0^{\infty} \psi(\tau) T(\tau)\, d\tau$ expected transmissibility gives the probability of disease transmission if we have no knowledge of removal time

Disease transmission dynamics

- $J(t)$ is rate of new infection, $J(t)dt$ gives the number of new infections between t and $t+dt$
- $N_i(t)=\int_0^t J(t-\tau)\Psi(\tau)d\tau$ is the total number of infectious individuals
- $N_r(t)=\int_0^t J(t-\tau)(1-\Psi(\tau))d\tau$ is the total number of removed individuals
- $N_s(t)=N-N_r(t)-N_i(t)$ is the total number of susceptible individuals

Disease transmission dynamics

- $R_0 = Z_x T$ is the basic reproduction number and it gives the expected number of infections a typical infected individual can cause
- We rewrite

$$R_0 = Z_x \int_0^\infty \psi(\tau) T(\tau) d\tau = Z_x \int_0^\infty \Psi(\tau) \frac{dT(\tau)}{d\tau} d\tau$$

- The rate of new infection caused with an infectious individual, by age of infection τ is then given by

$$J(\tau) = Z_x \Psi(\tau) \frac{dT(\tau)}{d\tau}$$

Disease transmission dynamics

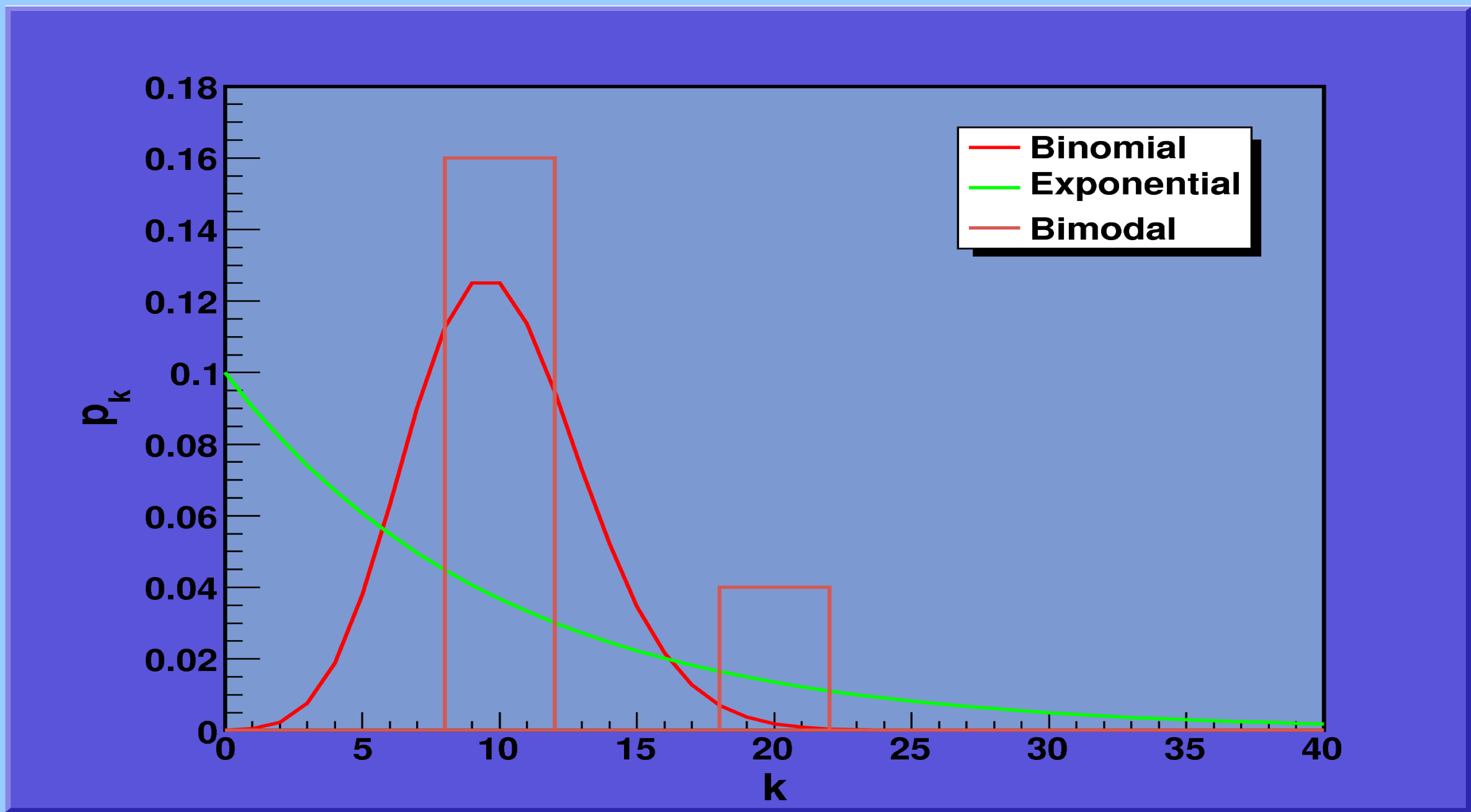
- The renewal equation takes the following form (when all individuals have the same degree and the system is infinite)

$$J(t)=\int_0^t \boxed{Z_x \Psi(\tau)\frac{dT(\tau)}{d\tau}} J(t-\tau)\,d\tau$$

- The renewal equation takes the following final form (for finite system and arbitrary degree distribution)

$$J(t)=\int_0^t J(t-\tau)\Psi(\tau)\frac{dT(\tau)}{d\tau} Z_x(\tau,t)\,d\tau$$

Networks types



Example 1:Poisson/binomial network

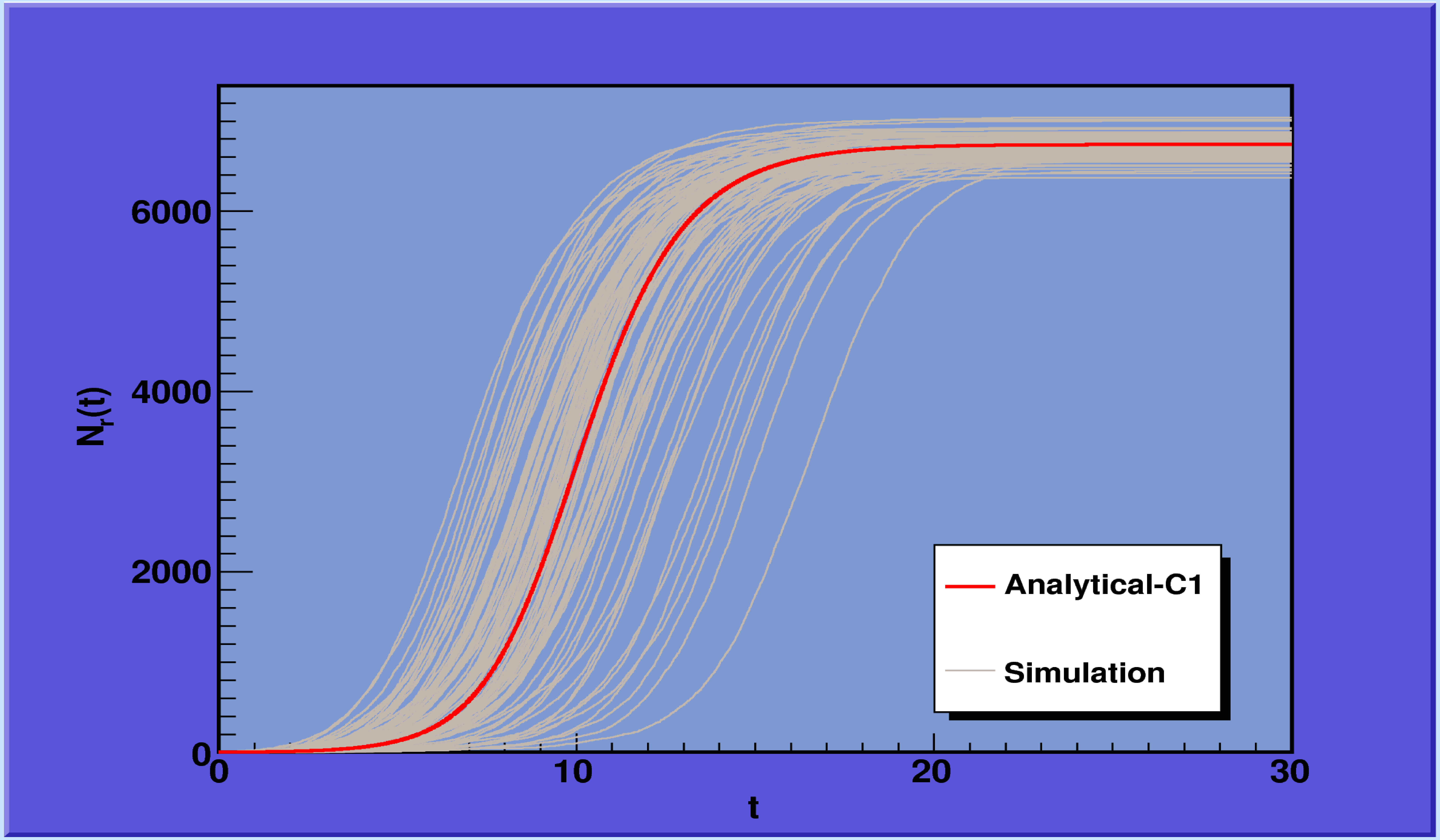
$$Z_x(\tau, t) = Z_x \frac{N_s(t)}{N} \qquad Z_x \frac{dT(\tau)}{d\tau} = \beta$$

$$J(t) = \beta \frac{N_s(t)}{N} \int_0^t J(t-\tau)\Psi(\tau)\, d\tau$$

$$\frac{dN_i(t)}{dt} = J(t) - \lambda_r N_i(t) = \beta \frac{N_s(t)}{N} N_i(t) - \lambda_r N_i(t)$$

J.Miller, B. Davoudi, R. Meza, A. Slim, B. Pourbohloul,
Journal of theoretical biology, 262 , 107 (2009)

Example 1:Poisson/binomial network



Disease transmission dynamics

- How does the excess degree of infected individuals change over time?

- What fraction of stubs of infected individuals is pointing to susceptible individuals?

Disease transmission dynamics

- The collection process:

1)We randomly choose individuals by targeting random stubs and then assign the individuals to collected group

2)We update the degree distributions of collected and uncollected individuals.

Disease transmission dynamics

$$n \qquad\qquad N-n$$

$$\tilde{p}_k(n) \qquad\qquad p_k(n)$$

$$\tilde{z}(n) \qquad\qquad z(n)$$

$$q_k(n)=\frac{kp_k(n)}{z(n)}$$

$$\frac{dp_k(n)}{dn}=\frac{p_k(n)}{N-n}\left(1-\frac{k}{z(n)}\right) \qquad \frac{d\,\tilde{p}_k(n)}{dn}=\frac{p_k(n)}{n}\left(\frac{k}{z(n)}-1\right)$$

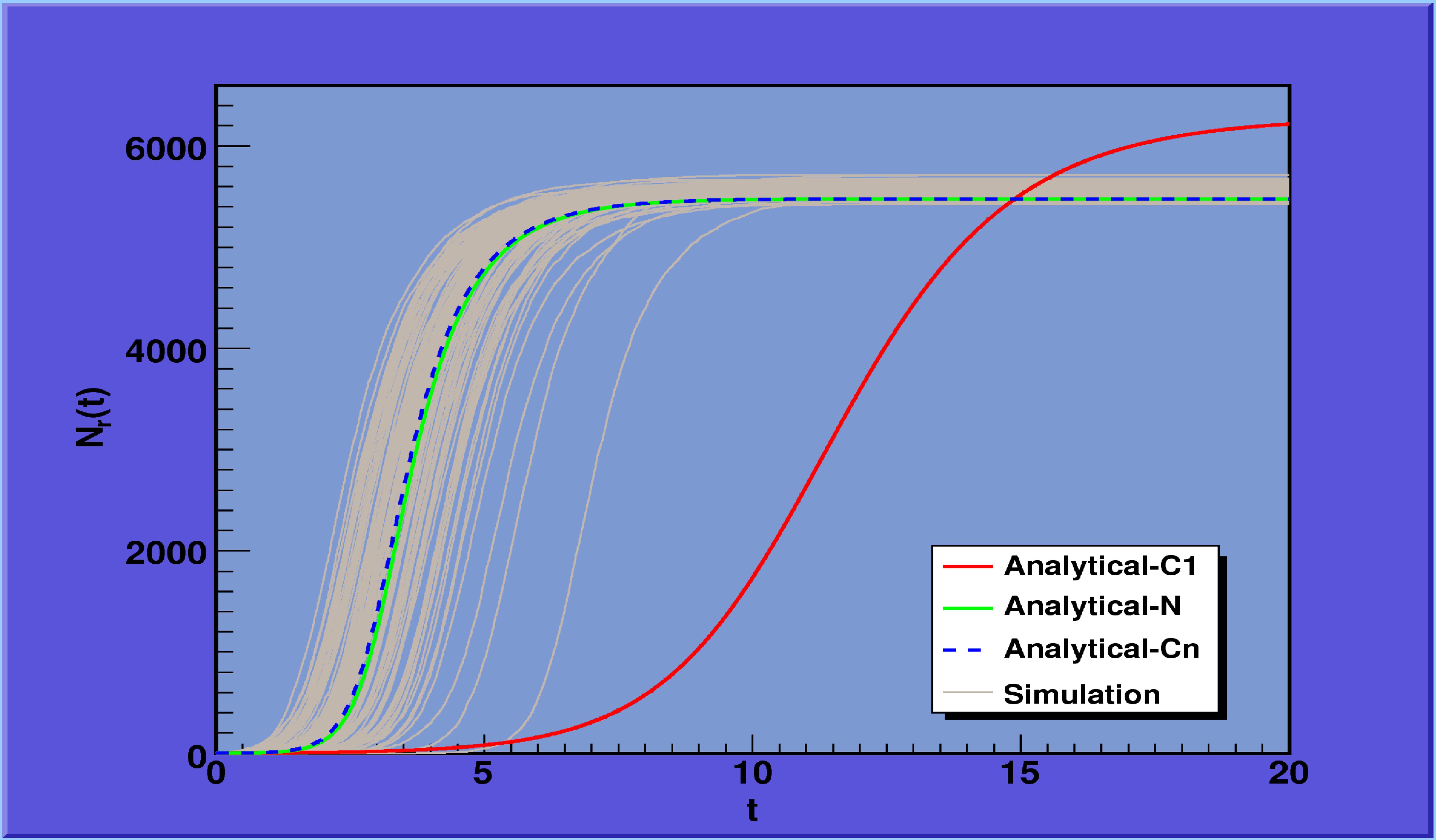
Disease transmission dynamics

- The average degree of susceptible individuals $z_s(t) = z(N_i(t) + N_r(t))$
- The average degree of infectious individuals that got infected during time t and $t+dt$

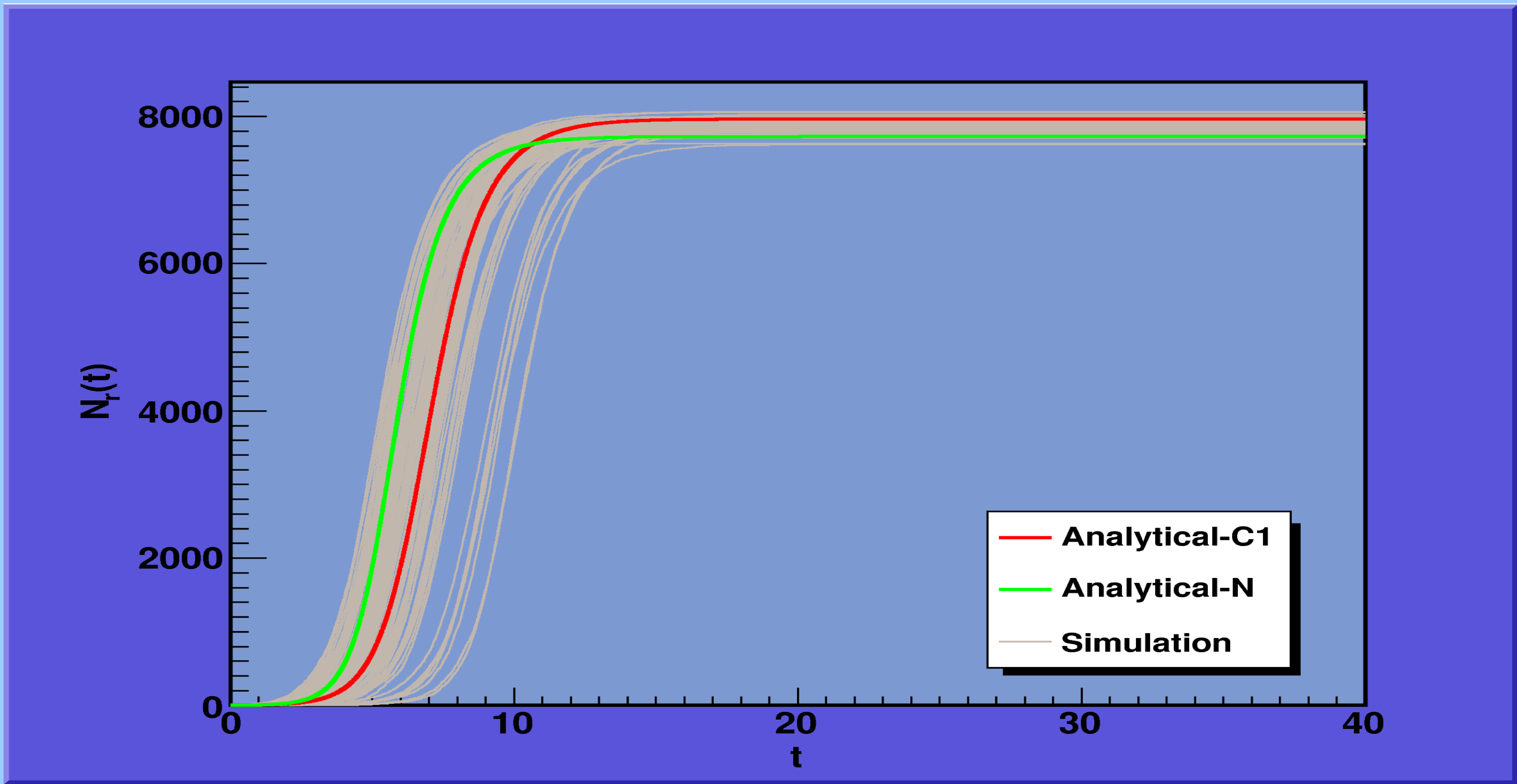
$$z_j(t) = \tilde{z}(N_i(t) + N_r(t)) + [N_i(t) + N_r(t)]\left[\frac{d\,\tilde{z}(n)}{dn}\right]_{n=N_i(t)+N_r(t)}$$

- Then $Z_x(\tau, t) = z_j(t-\tau)\dfrac{N_s(t)\,z_s(t)}{Nz}$

Example 1:exponential network



Example 1:bimodal network



Extensions

- Multi-types network
- Open network
- Dynamic network
- SIRS system (*etc*)

Conclusion

- We obtain the renewal equation

- We obtain a good approximation for the kernel of the renewal equation

- We test our results against simulation

- We discuss the possible extensions

Thank you

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