Quantum nonequilibrium steady states: an exact solution

Marko Žnidarič

Instituto de Ciencias Físicas
Universidad Nacional Autónoma de México, Cuernavaca, Mexico, and
Department of Physics
Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

“Symposium in honor of Francesco Calogero”, Cuernavaca
2010

Outline

Motivation

Equilibrium physics

- Well understood, canonical measure $\rho \sim \exp(-\beta H)$.
- A number of analytically solvable models.
- In 1d, symmetric systems, phase transitions only at $T = 0$.
- Thermal fluctuations destroy long-range order.

Nonequilibrium physics

- Unknown measure; does universal measure exist?
- Almost no solvable quantum models.
- Nonequilibrium phase transitions possible in 1d.
- Interesting new physics...?

Nonequilibrium physics

Known results

- Classical: many solvable stochastic models
  - Exclusion processes, reaction-diffusion systems... (revs.: Stinchcombe 2001; Derrida 2007)
  - Typical long-range order.
  - Measure is non-local (Derrida & Lebowits & Speer, 2001).
- Quantum: recent nonequilibrium solutions ($n \to \infty$)

System + environment: transient

- Star-system: (Breuer, Burgarth & Petruccione 2004)
- Quadratic XY model: (Araki & Ho 2000; Ogata 2002; Aschbacher & Pilet 2003; Platini & Karevski 2007)

...cont.

Known results (cont.)

- Quantum: recent nonequilibrium solutions ($n \to \infty$)

Master equation : stationary

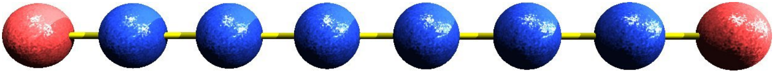
- Phase transitions in nonequilibrium steady state (NESS): quadratic systems, XX with dephasing, Heisenberg chain (Ljubljana group).

simplest situation: nonequilibrium stationary states

NESS

Real nonequilibrium situation: system + bath
(quenches are no good for NESS)

- 1d system
- Local coupling to baths at the boundaries



In NESS we can study, e.g., transport, presence of long-range order...

Diffusive (normal) transport

Fourier's law

$$j = -\kappa \nabla T$$

Microscopic origin not very clear

Jean Baptiste Joseph Fourier (1768-1830)

Is a given system diffusive or ballistic?

Heisenberg model $H = \sum \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z$

- One of the oldest quantum models (Ising suggested in '20 by Lenz).

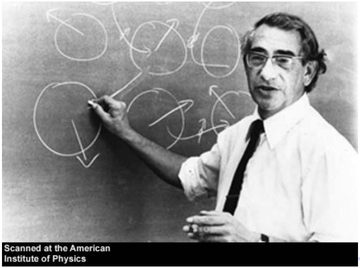
Zur Theorie des Ferromagnetismus.

Von **W. Heisenberg** in Leipzig.

Mit 1 Abbildung. (Eingegangen am 20. Mai 1928.)

- Special limit of the Hubbard model (John Hubbard '63).
- Even exactly solvable in 1d by Bethe ansatz.
- Spin conduction is controversial (normal or anomalous?).

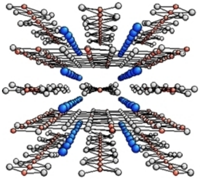
Heisenberg 



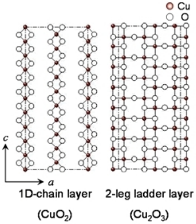
Hubbard

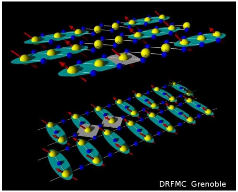
1d spin ladders and chains

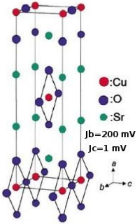
Superconducting spin-ladder compound $Sr_{14}Cu_{24}O_{41}$:

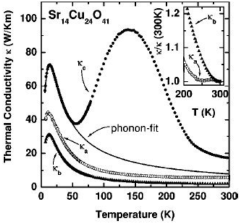


Akimitsu Lab.









- exchange interaction between spin-$\frac{1}{2}$ Cu
- 1d isotropic AF Heisenberg in Sr_2CuO_3
- explain $\approx 10^2$ larger magnetic contribution

Hess et al., PRB **64**, 184305 (2001)

How can we study transport - analytical

ballistic $\equiv j \sim n^0$ diffusive $\equiv j \sim 1/n$

Available methods

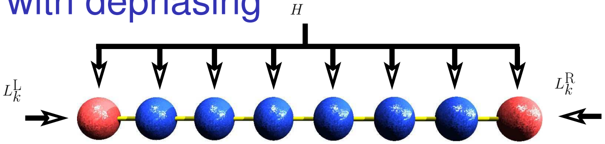
- Linear response formalism (Hamiltonian):
  - Quadratic systems (XY chain) : explicit diagonalization; all ballistic
  - Bethe ansatz (Heisenberg) : ?
  - Rigorous Mazur's inequality (ballistic transport)
- True nonequilibrium (Open system; master equation):
  - Quadratic Lindblad generator : ballistic (Prosen et.al.; Karevski&Platini)

We want to understand when is it diffusive, but no solvable diffusive known!

What is our new contribution?

- New solvable model: master equation solvable for any n.
- The only known solvable diffusive model.
- Interesting nonequilibrium physics.

XX chain with dephasing



$$H = \sum_{j=1}^{n-1} (\sigma_j^{\mathrm{x}} \sigma_{j+1}^{\mathrm{x}} + \sigma_j^{\mathrm{y}} \sigma_{j+1}^{\mathrm{y}})$$

Lindblad master equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = \mathrm{i}[\rho, H] + \mathcal{L}^{\mathrm{bath}}(\rho) + \mathcal{L}^{\mathrm{deph}}(\rho) = \mathcal{L}(\rho). \quad (1)$$

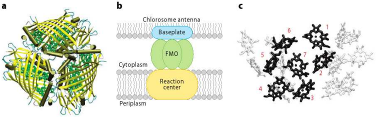
$$\mathrm{Bath}: \quad L_1^{\mathrm{L,R}} = \sqrt{\Gamma} \frac{\sqrt{1 \mp \mu}}{2} \sigma_{1,n}^{+}, \quad L_2^{\mathrm{L,R}} = \sqrt{\Gamma} \frac{\sqrt{1 \pm \mu}}{2} \sigma_{1,n}^{-}. \quad (2)$$

Dephasing, $\mathcal{L}^{\mathrm{deph}} = \sum_{i=1}^{n} \mathcal{L}_i^{\mathrm{deph}}$

$\mathcal{L}_i^{\mathrm{deph}}$ of Lindblad form with one $L = \sqrt{\gamma/2}\,\sigma_i^{\mathrm{z}}$:

$$\mathcal{L}_i^{\mathrm{deph}}(\rho) = [L\rho, L^\dagger] + [L, \rho L^\dagger]. \tag{3}$$

- $\mathcal{L}_i^{\mathrm{deph}}(\sigma^{\mathrm{x,y}}) = -2\gamma\sigma^{\mathrm{x,y}}$ and $\mathcal{L}_i^{\mathrm{deph}}(\sigma^{\mathrm{z}}) = 0$.
- Exponential decay of off-diagonal elements.
- E.g., due to coupling to external DOF (at high T)
- Considerable recent attention:
  - Transport in short networks: light-harvesting complex (Aspuru-Guzik et al.; Plenio et.al.). Dephasing can enhance transport



  - Heisenberg model (Žnidarič, NJP '10)

Ansatz for XX with dephasing

$$\rho \sim \mathbb{1} + \mu(A+B) + \frac{\mu^2}{2}(AB+BA) + \mu^2(C+D+F) + \mathcal{O}(\mu^3), \quad (4)$$

$$\mu A = \sum_{j=1}^{n} a_j \sigma_j^{\mathrm{z}}, \qquad \mu B = \frac{b}{2} \sum_{k=1}^{n-1} j_k, \quad (5)$$

$$\mu^2 C = \sum_{j=1}^{n} \sum_{k=j+1}^{n} (C_{j,k} + a_j a_k) \sigma_j^{\mathrm{z}} \sigma_k^{\mathrm{z}}, \quad (6)$$

$$\mu^2 D = \sum_{j=1}^{n-2} \frac{d_j}{2} \left(\sum_{l=j+1}^{n-1} \sigma_j^{\mathrm{z}} j_l - \sum_{l=1}^{n-1-j} j_l \sigma_{n+1-j}^{\mathrm{z}} \right), \quad (7)$$

$$\mu^2 F = \frac{f}{8} \sum_{\substack{k,l=1 \\ k \neq l}}^{n-1} j_k j_l. \quad (8)$$

Ansatz (Žnidarič, JSTAT '10, arXiv:1011.0998)

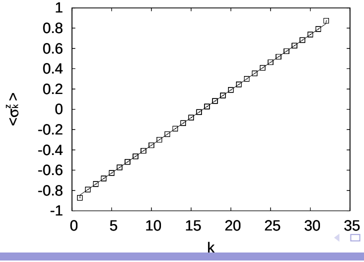
- Expansion in powers of the driving μ.
- Linear terms A and B: magnetization profile and current.
- Quadratic terms: correlations current-spin, spin-spin, current-current.

Procedure

- Write equations for unknown coefficients, like magnetization at site i.
- These equations are exact (to all orders in μ).
- Consequences:
  - All terms from the ansatz are calculated exactly!
  - All terms linear, quadratic and of μ^3 are calculated exactly.
  - All single, two-point and three-point correlations are exact!

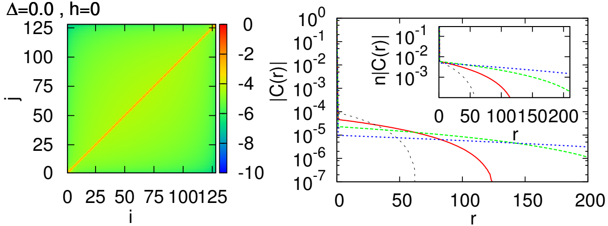
Diffusive: $\gamma \neq 0$

- The only solvable diffusive model.
- Linear magnetization profile (typical for diffusive).
- Current $j \sim 1/n$; normal (diffusive) spin transport.
- Long-range correlations: (purely nonequilibrium effect)
  - $C(x,y) = -\frac{(2\mu)^2}{n} x(1-y)$ (solution of Laplace eq.).
  - At fixed distance plateau $C \sim j \cdot \mu \sim 1/n$.
  - Not Gaussian.



...cont.

For $\gamma \neq 0$ long-range correlations $C(x,y) = -\frac{(2\mu)^2}{n} x(1-y)$



Ballistic: $\gamma = 0$

- Known XX model (Karevski&Platini PRL '09).
- Flat magnetization profile (typical for ballistic).
- Current constant, $j \sim n^0$.
- No long-range correlations; Gaussian.

MPO form for $\gamma = 0$ (Žnidarič JPA '10)

- Nonequilibrium phase transition at $\gamma = 0$.
- $\gamma = 0$ especially simple: ballistic, no long-range correlations.
- Compact exact solution possible:

$$\rho = \frac{1}{2^n} \sum \langle 1 | A_1^{s_1} A_2^{s_2} \cdots A_n^{s_n} | 1 \rangle \, \sigma_1^{s1} \cdots \sigma_n^{s_n}$$

Exact MPO solution

- $A_i = (A_i^{(\mathrm{x})}, A_i^{(\mathrm{y})}, A_i^{(\mathrm{z})}, A_i^{(\mathbb{1})})$
- Divergence $\mathcal{L}_{i,i+1}^{(\mathrm{H})}(A_i \otimes A_{i+1}) = A_i \otimes M_{i+1} - M_i \otimes A_{i+1}$.
- $D \times D$ state-matrices A_i and auxiliary M_i.

MPO

Dimension of A_i is $D = 4$ and independent of chain length.

$$A_i^{(z)} = \begin{pmatrix} a_i & -t^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_i^{(\mathbb{1})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & t & 0 \end{pmatrix},$$

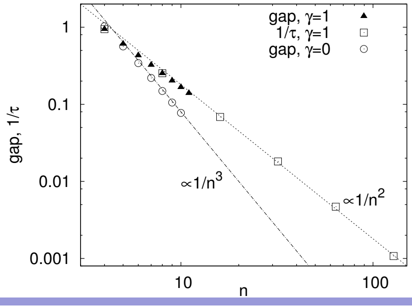
$$A^{(x)} = (-P, -P, P, P, -P, \ldots), \qquad A^{(y)} = (-R, R, R, -R, -R, \ldots),$$

$$P = \begin{pmatrix} 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

Nonequilibrium phase transitions

How to define noneq. phase transition?

- Concept of a partition function (zeroes): does not exist!
- In our work observables change: long vs. short range correlations.
- What about the gap of the Liouville superoperator?



- Gap always decreases with n!
- At the phase transition the gap decays faster!

Other models

We began motivation with the Heisenberg model.

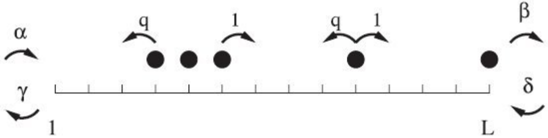
Heisenberg model (Prosen & Žnidarič '09,'10)

- No analytical solution (so far).
- Numerics:
  - Normal diffusive transport at $T = \infty$.
  - Long-range correlations for $\Delta > \Delta_c \approx 0.91$!
  - Nonequilibrium phase transition.
  - Correlation plateau as $C \sim (\Delta\mu)^2$ (independent of n!).

Classical lattice gasses

Solution for quantum XX with dephasing is similar to solutions of classical stochastic models.

- Very rich physics.
- ASEP solution in '93 (Derrida et.al.).



- Nonequilibrium phase transitions.
- Explicit MPO solutions.

Is there a deeper connection?

Summary

1. XX with dephasing:
   - 1d XX chain coupled to Lindblad at ends.
   - Each spin experiences dephasing.
   - Exact solution (one-, two- and three-point functions).
   - Long-range order.
   - Non-gaussian.
   - Nonequilibrium phase transition at $\gamma = 0$.
   - Only known solvable diffusive model.
   - MPO solution with $D = 4$ in the limit $n \to \infty$ and $\mu \to 0$.
2. Without dephasing:
   - Simple solution in terms of MPO; Gaussian.
   - Matrices of fixed dimension $D = 4$ irrespective of n.
   - Similarity to classical stochastic processes.