The Riemann-Hilbert approach to higher dimensional integrable systems

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Outline

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The Riemann Hilbert problem

- **1.1** [Riemann, 1851]
  Let $\{M_t\}_{t\in\mathbb{S}^1}$ be a family of curves in $\mathbb{C}$. Find all functions ω holomorphic in the open unit disk $\mathbb{D}$ and

$$\omega(t)\in M_t,\quad t\in\mathbb{S}^1. \tag{1}$$

- **1.2** [Hilbert, 1904]
  Write $\omega(t)=u(t)+iv(t)$, and (1) as

$$F(t,u(t),v(t))=0 \tag{2}$$

for a given $F:\mathbb{S}^1\times\mathbb{R}^2\to\mathbb{R}$. Differentiating the equation w.r.t. t, we obtain

$$a(t)\tilde{u}(t)+\beta(t)\tilde{v}(t)=\gamma(t),\quad \forall t\in\mathbb{S}^1,$$

or

$$\Phi^+(t)-g(t)\Phi^-(t)=f(t),\quad t\in\mathbb{S}^1 \tag{3}$$

for given functions $g(t)$ and $f(t)$.

- **1.3 Theorem** (Scalar RH problem) [Plemelj, 1908], [Gakhov, 1938]
  The solution of the scalar homogeneous RH problem (3) with Holder continuous g is given by

$$\Phi(z) = X(z)P_{m+\kappa},$$

$$X(z) = \begin{cases} e^{\Gamma(z)}, & |z| < 1 \\ z^{-\kappa} e^{\Gamma(z)}, & |z| > 1 \end{cases}$$

$$\Gamma(z) = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{\log(\tau^{-\kappa} g(\tau))}{\tau - z} d\tau,$$

  where $P_{m+\kappa}$ is a polynomial of degree $m+\kappa$, and $\kappa = \operatorname{index} g(t)$.

The Riemann Hilbert problem

- **1.5 Singular integral** [Carleman, 1922]
  Define the Cauchy integral

$$\mathcal{C}f(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-z}\,dt.$$

  Suppose the function $f \in H^1(\mathbb{R})$. Then
  - $\mathcal{C}f \in H^1(\mathbb{R})$, $\mathcal{C}f(z)$ *is holomorphic and bounded in* $\mathbb{C}\backslash\mathbb{R}$;
  - $\mathcal{C}f$ *is uniformly Holder continuous of order* $1/2$ *on* $\mathbb{C}\backslash\mathbb{R}$;
  - $(\mathcal{C}f)^{\pm}(t)$ *exists pointwise for* $t \in \mathbb{R}$ *and* $\mathcal{C}f \to 0$ *uniformly as* $z \to \infty$.

- **1.5 Small data matrix RH problem** [Beals-Coifman, 1984]
  Suppose g satisfies

$$|g-1|_{H^1(\mathbb{R},dt)} << 1.$$

Then there exist $\Phi \in L^\infty(\mathbb{R}, dt) + H^1(\mathbb{R}, dt)$ *such that*

$$\begin{aligned} \Phi^+(t) = g(t)\Phi^-(t), &\qquad t \in \mathbb{R}, \\ \Phi(\lambda) \to 1, &\qquad z \to \infty, \\ \Phi(z) \textit{ is holomorphic} &\qquad z \in \mathbb{C}/\mathbb{R}, \end{aligned}$$

bounded and absolutely continuous in z.

The Riemann Hilbert problem

- **1.6 Large data matrix RH problem** [Beals-Coifman, 1984] *Suppose*
  - $$g(t) = (1 + U(t))D(t)(1 + L(t)), \quad \forall t \in \mathbb{R}$$
    where U upper triangular, D diagonal, L lower triangular and U, $D-1$, $L \in H^1(\mathbb{R}, dt)$,
  - *The homogeneous RH problem (w. 0 at infinity) has only the trivial solution.*

  Then there exist $\Phi \in L^\infty(\mathbb{R}, dt) + H^1(\mathbb{R}, dt)$ such that

$$\begin{aligned} \Phi^+(t) = g(t)\Phi^-(t), &\qquad t \in \mathbb{R}, \\ \Phi(\lambda) \to 1, &\qquad z \to \infty, \\ \Phi(z) \text{ is holomorphic} &\qquad z \in \mathbb{C}/\mathbb{R}, \end{aligned}$$

bounded and absolutely continuous in λ.

Outline

- **2.1 Integrable systems** [Kruskal, Zabusky, Greene, Gardner, Miura, 1967], [Lax, 1968], [Beals, Coifman, 1985]
  An integrable system of $q(\mathbf{x}, t)$ is a nonlinear evolution equation which can be written as

$$[L_\lambda, M_\lambda] = 0$$

  and the initial value problem of the integrable system can be solved by using the inverse scattering theory of

$$L_\lambda \Psi(\mathbf{x}, t, \lambda) = 0.$$

  That is,

$$\begin{array}{ccccc} q(\cdot, 0) & \longleftrightarrow & L_\lambda(0) & \longrightarrow & S[L_\lambda(0)] \\ \downarrow & & & & \downarrow \\ q(\cdot, t) & \longleftrightarrow & L_\lambda(t) & \longleftarrow & S[L_\lambda(t)] \end{array}$$

- **2.2 Direct problem**

  Given $q(\mathbf{x}, t)$, by

$$[\bar{\partial}_\lambda, L_\lambda] = 0, \tag{4}$$

  the eigenfunction $\Psi(\mathbf{x}, t, \lambda)$ constructed is holomorphic outside contours $\Sigma \subset \mathbb{C}$. Thus Ψ, or q, can be determined by $\bar{\partial}_\lambda \Psi$ which is supported on Σ.

  Moreover, (4) implies $L_\lambda\left(\bar{\partial}_\lambda \Psi\right) = 0$, hence $\bar{\partial}_\lambda \Psi$ can be written as a multiple or an integral operator of $\Psi(\mathbf{x}, t, \lambda)$, where the multiple or the kernel is furthermore characterized by the algebraic properties of L_λ. Therefore, we choose the scattering data S_λ to be the essential part of the multiple or the kernel.

- **2.3 Inverse problem**

  Since the scattering data characterizes the $\bar{\partial}$-data, the eigenfunction Ψ and its asymptotic properties can be found by solving a RH problem.

  To find the potential $q(\mathbf{x}, t)$, note the property (4) ($\left[\bar{\partial}_\lambda, L_\lambda\right] = 0$) also implies $\bar{\partial}_\lambda \left(L_\lambda \Psi\right) = 0$. Together with the algebraic properties of S_λ, we have

  $$\Psi \textit{ and } L_\lambda \Psi \textit{ satisfy the same RH problem.}$$

  By investigating the λ-asymptotic property of $L_\lambda \Psi$, one can determine the potential $q(\mathbf{x}, t)$.

Outline

Goal

We illustrate the ideas in solving the following higher dimensional integrable systems via the RH approach.

- The direct and inverse scattering problem of the Ward equation with **large** initial data and **purely continuous** scattering data;
- The inverse scattering problem of a twisted hierarchy associated with the generalized sine-Gordon equation.

Outline

4.1 The Ward equation

Taking a dimension reduction and a gauge fixing of the self-dual Yang-Mills equation in the space-time with signature $(2,2)$, one derives a $2+1$ dimensional $SU(N)$ chiral field equation with an additional torsion term.

$$-\left(J^{-1}J_t\right)_t+\left(J^{-1}J_x\right)_x+\left(J^{-1}J_y\right)_y+\nu_0\left\{\left(J^{-1}J_y\right)_x-\left(J^{-1}J_x\right)_y\right\}$$
$$+\nu_1\left\{\left(J^{-1}J_t\right)_y-\left(J^{-1}J_y\right)_t\right\}+\nu_2\left\{\left(J^{-1}J_x\right)_t-\left(J^{-1}J_t\right)_x\right\}=0.$$

Where J lies in $SU(N)$ and $\nu=(\nu_0,\nu_1,\nu_2)$ is a constant unit vector. Letting $\nu=(1,0,0)$ (time-like) and $\nu=(0,1,0)$ (space-like), we obtain two integrable systems, the 3-dimensional relativistic-invariant system and the Ward equation.

The Ward equation

- **4.2 Previous works**

  The construction of simple solitons, and the study of their scattering properties was done by [Manakov, Zakharov, 81] for the 3-dimensional relativistic-invariant system and by many mathematicians for the Ward equation, see [Dai, Terng, 07] for references.

  If the initial potential is sufficiently small, [Manakov, Zakharov, 81], [Villarroel, 89] studied the inverse scattering problem and solve the Cauchy problem of the 3-dimensional relativistic-invariant system and [Villarroel, 90], [Fokas, Ioannidou, 01], [Dai, Terng, Uhlenbeck, 06] of the Ward equation.

The Ward equation

- **4.2 Our results**
  We solve the Cauchy problem of the Ward equation with large (and purely continuous) data by IVS in [Wu,08] and with mixed scattering data via a theory of Backlund transformation in [Wu, 09].
  The Lax pair is derived as follows

$$(J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y - [J^{-1}J_t, J^{-1}J_y] = 0$$

Then

$$\begin{gathered}[\lambda\partial_x - \partial_\xi - A, \lambda\partial_\eta - \partial_x - B] = 0, \\ \xi = -\frac{t+y}{2}, \quad \eta = \frac{t-y}{2}, \\ J(x,y,t,0) = \Psi(x,y,t,0)^{-1}, \\ A = -\frac{\partial\Psi}{\partial\xi}\Psi^{-1}, \ B = -\frac{\partial\Psi}{\partial x}\Psi^{-1}.\end{gathered}$$

The Ward equation

- **4.2 Our results**(continued) Set

$$A=-\partial_x Q,\ \ B=-\partial_\eta Q$$
$$(\eta,x,\xi)\to(x,y,t),$$

then the Ward equation turns into

$$\partial_x\partial_t Q=\partial_y^2 Q+[\partial_y Q,\partial_x Q]$$

with a Lax pair

$$(\partial_y-\lambda\partial_x)\,\Psi=(\partial_x Q)\Psi,$$
$$(\partial_t-\lambda\partial_y)\,\Psi=(\partial_y Q)\Psi.$$

The Ward equation

- **4.2 Our results**(continued)Three important algebraic properties of the Lax operator $L_\lambda = \partial_y - \lambda\partial_x - q$ of the Ward equation,

  (1) derivation property;

  (2) translating invariant property;

  (3) the principal part being equivalent to $\bar\partial$,

  are used to reformulate the direct problem as a RH problem (hence the large data difficulty is resolved). More precisely,

- **4.2 Our results**(continued)
  - Translating invariant property:

$$\textit{If } \Psi(x,y,\lambda) \textit{ is an e.f. of } q(x,y),$$
$$\textit{then } \Psi(x,y+y_0,\lambda) \textit{ is an e.f. of } q(x,y+y_0).$$

Hence wlog, we can assume

$$\begin{aligned} q &= q^- + q^+, \\ q^- &= 0,\ \textit{for } y \geq 1, \\ q^+ &= 0,\ \textit{for } y \leq -1, \\ |\hat{q}_x^{\pm}(\xi,y)|_{L_1(d\xi dy)} &< \left(\frac{3}{2}\right)^N, \end{aligned}$$

and use an induction scheme to find the eigenfunctin Ψ. Since

- **4.2 Our results**(continued)
  - Using the derivation property of $\partial_y - \lambda\partial_x$, any eigenfunction Ψ for q, whenever it exists, must be of the form

$$\Psi(x,y,\lambda) = \begin{cases} \Psi^-(x,y,\lambda)a^-(x,y,\lambda), & y \le 0, \\ \Psi^+(x,y,\lambda)a^+(x,y,\lambda), & y \ge 0, \\ (\partial_y - \lambda\partial_x)a^\pm = 0. \end{cases}$$

    Here $(\partial_y - \lambda\partial_x - q^\pm)\Psi^\pm = 0$.
  - By a change of variables $x + \lambda y = \tilde{x} + i\tilde{y} = z$, $\tilde{x}, \tilde{y} \in \mathrm{R}$. The existence of Ψ is equivalent to solving the RH problem

$$\begin{aligned} &\partial_{\bar{z}} f = 0, \ \text{for } z \in \mathrm{C}^\pm, \\ &f^+ = f^-(F - 1), \\ &F = \left(\Psi^+\right)^{-1}\Psi^-. \end{aligned}$$

Hence the direct, and the inverse problems are reduced to solving two types of RH problem with large data.

- **4.3 Remarks**
  - Triangular factorization in the approximation process is necessary to reduce the linear system to be just-determined.
  - Need to solve RH problems with data composed of eigenfunctions derived by induction hypothesis.
  - Nice asymptote is required to assure the solvability of the system of linear equations at infinity.
  - For the direct problem, solvability of the linear system outside a bounded discrete set in $\mathbb{C}\backslash\mathbb{R}$ is assured by the mero. property.
  - For the inverse problem, global solvability is assured by applying the **reality condition**.

Outline

The twisted hierarchy

- **5.1 The sine-Gordon equation**
  - $$\mathcal{P} = \left\{ M \text{ surfaces of } -1 \text{ curv. in } \mathbb{R}^3 \right\}$$
    $$\longleftrightarrow \{u : u_{xx} - u_{tt} = \sin u\} \qquad (SGE)$$
  - [Backlund, 1875]
    - If $M \equiv \overline{M}$, then $M, \overline{M} \in \mathcal{P}$.
    - If $M \in \mathcal{P}$, then $\exists M_\lambda \in \mathcal{P}$, s.t. $M \equiv M_\lambda$.
  - [Ablowitz, Kaup, Newell, Segur, 73]
    Lax pair & IVS of (SGE)

- **5.2 The generalized sine-Gordon equation**
  - [Chern, Terng, Tenenblat, 80]
    - $$\mathcal{P} = \left\{ M^n \text{ submfld of } -1 \text{ sec. curv. in } \mathbb{R}^{2n-1} \right\}$$
      $$\longleftrightarrow \{\text{solutions of } (GSGE)\ \}$$
    - (1) If $M \equiv \overline{M}$, then $M, \overline{M} \in \mathcal{P}$.
      (2) If $M \in \mathcal{P}$, then $\exists M_\lambda \in \mathcal{P}$, s.t. $M \equiv M_\lambda$.
    - $I = \sum_{i=1}^n \alpha_{1i}^2\, dx_i^2$, $II = \sum_{i=2,j=1}^n \alpha_{ij}\alpha_{1j}\, dx_j^2 e_{n+i-1}$.

$$\begin{aligned} & \alpha \in O(n), \\ \partial_{x_j}\alpha_{ki} &= \alpha_{kj}h_{ji},\ h_{ii} = 0, && i \neq j, \\ \partial_{x_i}h_{ij} + \partial_{x_j}h_{ji} + \textstyle\sum_{k\neq i,\,j} h_{ki}h_{kj} &= \alpha_{1i}\alpha_{1j}, && i \neq j \\ \partial_{x_k}h_{ij} &= h_{ik}h_{kj}, && i,j,k \ \ distinct \end{aligned}$$

The twisted hierarchy

- **5.2 The generalized sine-Gordon equation** (continued)
  - [Ablowitz, Beals, Tenenblat,86]
    Lax pair & IVS of (GSGE)
  - [Terng, 10]
    Twisted U/K-hierarchies,
    Twisted $O(n,n)/O(n)\times O(n)$-system (GSGE),
    Local solutions via a loop group approach
- **5.3 The generating equation** [Campos, Tenenblat, 94]

$$\mathcal{P} = \{ \text{Riem. } n\text{-submfld of } -1 \text{ sec. curv. in } \mathbb{R}^{2n-2,1} \}$$
$$\leftrightarrow \{ \text{solutions of (GE)} \}$$
$$\alpha \in O(n-1,1),$$
$$\partial_{x_j}\alpha_{ki} = \alpha_{kj}h_{ji},\ h_{ii} = 0,\ i \neq j,$$
$$\partial_{x_i}h_{ij} + \partial_{x_j}h_{ji} + \sum_{k\neq i,j} h_{ki}h_{kj} = \alpha_{1i}\alpha_{1j},\ i \neq j$$
$$\partial_{x_k}h_{ij} = h_{ik}h_{kj},\ i,j,k \ \ distinct$$

- **5.3 The generating equation** (continued) They constructed the Backlund transform and linearized it but could not solve the direct and inverse scattering problem.
- **5.4 Twisted $\frac{O(J,J)}{O(J)\times O(J)}$-hierarchies** [Ma, Wu]
  - Let $J = I_{n-q,q} = \mathrm{diag}(\overbrace{1,\cdots,1}^{n-q \text{ times}}, \overbrace{-1,\ldots,-1}^{q \text{ times}})$, $\tilde{J} = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}$, and

$$\begin{aligned} O(J,J) &= \left\{ x \in GL_{2n}(\mathbb{R}) | \; \tilde{J} x^t \tilde{J} x = 1 \right\}, \\ L(O(J,J)) &= \{ f : \mathbb{S}^{1/\epsilon} \to GL_{2n}(\mathbb{R}), \; \tilde{J} f\left(\bar{\lambda}\right)^* \tilde{J} f(\lambda) = 1 \}. \end{aligned}$$

- **5.4 Twisted $\frac{O(J,J)}{O(J)\times O(J)}$-hierarchies** (continued)
  - Let σ_i, $i \in \{0, 1\}$, be the involutions on $O(J, J)$ defined by

$$\sigma_i(x) = I_{n+i,n-i} x I_{n+i,n-i}^{-1}$$

$$and \quad \hat\sigma_0(f)(\lambda) = \sigma_0(f(-\lambda)), \;\; \hat\sigma_1(f)(\lambda) = \sigma_1(f(\frac{1}{\lambda})),$$

be involutions on $L(O(J, J))$. Let

$$L^{\sigma_0} = \{f \in L(O(J, J)) | \; \hat\sigma_0(f) = f\} \subset L(O(J, J)).$$

Then there are splittings of the loop group L^{σ_0} given by the subgroups

$$L_+^{\sigma_0} = \{f \in L^{\sigma_0} | f : \mathbb{A}_{\epsilon,1/\epsilon} \overset{holo.}{\rightarrow} GL_{2n}(\mathbb{R}), \, \hat\sigma_1(f) = f, \; f(1) \in S_0\},$$
$$L_-^{\sigma_0} = \{f \in L^{\sigma_0} | f : \mathbb{C}/\mathbb{D}_{1/\epsilon} \overset{holo.}{\rightarrow} GL_{2n}(\mathbb{R}), \; f(\infty) \in K_0'\}.$$

- **5.4 Twisted $\frac{O(J,J)}{O(J)\times O(J)}$-hierarchies** (continued)
  - Let $\hat{\pi}_\pm$ denote the projections onto $\mathcal{L}_\pm^{\sigma_0}$ with respect to the splitting and
    $$J_{a,j} = a\lambda^j + \sigma_1(a)\lambda^{-j} \in \mathcal{L}_+^{\sigma_0}$$
    for some constant regular $a \in \mathcal{A}$. Here $\mathcal{A}$ be a maximal abelian subalgebra of $\mathcal{P}_0$, $o(J,J) = \mathcal{K}_0 + \mathcal{P}_0$.
  - **Definition:** The $2j+1$-th twisted $\frac{O(J,J)}{O(J)\times O(J)}$-flow (twisted by σ_1) is the compatibility condition of
    $$\left[\partial_x + \hat{\pi}_+\left(MJ_{a,1}M^{-1}\right), \partial_t + \hat{\pi}_+\left(MJ_{\tilde{a},2j+1}M^{-1}\right)\right] = 0,$$
    for some $M = M(x,\lambda) \in L_-^{\sigma_0}$.

The twisted hierarchy

- **5.4 Twisted $\frac{O(J,J)}{O(J)\times O(J)}$-hierarchies** (continued)
  - The associated Lax pairs are Laurent polynomials in the spectral variable λ. The reality conditions of the Lax pairs are induced from the simplest Dihedral group.
- **5.5 Twisted $\frac{O(J,J)}{O(J)\times O(J)}$-system**
  - **Definition:** The twisted $\frac{O(J,J)}{O(J)\times O(J)}$-system is the compatibility condition of

$$\left[\partial_{x_i}+\hat{\pi}_+\left(MJ_{a_i,1}M^{-1}\right),\partial_{x_j}+\hat{\pi}_+\left(MJ_{\tilde{a}_j,1}M^{-1}\right)\right]=0,$$

for some $M=M(x_1,\cdots,x_n,\lambda)\in L_-^{\sigma_0}$, $1\le i,j\le n$ and

$$a_i=\begin{pmatrix}0 & e_i\\ e_i & 0\end{pmatrix},\ e_i=diag(0,\cdots,0,\overbrace{1}^{i\text{-th entry}},0,\cdots,0).$$

- **5.6 Our results** [Ma, Wu]
  - The $2J+1$-th twisted $\frac{O(J,J)}{O(J)\times O(J)}$-flow is a nonlinear $2j+1$-th order partial differential systems in the coefficients of $\hat{\pi}_+\left(MJ_{a,1}M^{-1}\right)$. We obtain the hyperbolic sinh-Gordon equation and are working on other explicit formula for large j.
  - We have proved a global existence theorem of the initial value problem of the twisted $\frac{O(J,J)}{O(J)\times O(J)}$-flows (with mixed scattering data with arbitrary poles and multiplicity for $J=I$, and small data for $J\neq I$).
  - We discover the associated submanifold geometry: n-dimensional time-like submanifolds of constant negative sectional curvature in the $(2n-1)$-dimensional pseudo-Riemannian manifold of constant sectional curvature with signature $(2n-2,1)$ describe the geometry of the twisted $\frac{O(J,J)}{O(J)\times O(J)}$-systems, with $J=diag\,(1,\cdots,1,-1)$.

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