Holomorphic dynamical systems whose orbit spaces give new examples of compact complex manifolds

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Consider the linear differential equation in $\mathbb{C}^n$

$$\frac{dz_1}{dT} = \lambda_1 z_1$$

$$\vdots$$

$$\frac{dz_n}{dT} = \lambda_n z_n$$

We will always consider complex time $T \in \mathbb{C}$.

This equation can be written in matrix form:

$$\frac{dZ}{dT} = \Lambda Z,$$

where $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$ is adiagonal matrix, and $Z = (z_1, \cdots, z_n)^t$ is a column vector.

We will always assume that $\lambda_i \neq 0$ $(i = 1, \cdots, n)$ so that Λ is invertible. The equation determines a linear action of $\mathbb{C}$ on $\mathbb{C}^n$, in other words complex flow, $\varphi_T : \mathbb{C}^n \to \mathbb{C}^n$ (Thus, $\varphi_{T_1+T_2} = \varphi_{T_1} \circ \varphi_{T_2}$).

This linear flow is given explicitly by the formula:

$$\varphi_T(Z) = e^{T\Lambda} Z = (e^{T\lambda_1} z_1, \cdots, e^{T\lambda_n} z_n)^t \, , T \in \mathbb{C}, \, Z^t \in \mathbb{C}^n.$$

If $Z \neq 0$ the orbit or leaf of Z, denoted by $L(Z)$ is parametrized by the map from $\mathbb{C}$ to $\mathbb{C}^n$ given by the function

$$T \mapsto (e^{T\lambda_1} z_1, \cdots, e^{T\lambda_n} z_n)^t, \textit{ where } Z = (z_1, \cdots, z_n)^t.$$

The complement of the origin $\mathbb{C}^n - \{0\}$, is foliated by the orbits of the flow. The orbits are immersed copies of $\mathbb{C}$ or $\mathbb{C}^*$.
There is a dichotomy:

1. The origin of $\mathbb{C}$ is in the convex hull of $\{\lambda_1, \cdots, \lambda_n\}$.
   This happens if and only if there exists a point Z such that its leaf $L(Z)$ does not accumulate at the origin.
2. The origin of $\mathbb{C}$ is *not* in the convex hull of $\{\lambda_1, \cdots, \lambda_n\}$.
   This happens if and only if *every* leaf accumulates at the origin.

When 2) happens there exists (*generically*) an open set $\mathcal{U}$, saturated by the leaves of the flow, such that the space of leaves is a Hausdorff space which is in fact a complex manifold M. Even more, there is a free holomorphic action of $\mathbb{C}^*$ on M such that the orbit space is a compact complex manifold. The set $\mathcal{U}$ is the union of subspaces of $\mathbb{C}^n$ spanned by certain nonempty subsets of the canonical basis of $\mathbb{C}^n$.

BASIC EXAMPLE

In $\mathbb{C}$ consider a non-degenerate triangle with vertices λ_1, λ_2 and λ_3. Suppose that the origin is in the interior of this triangle. Then the open set $\mathcal{U} \subset \mathbb{C}^3$ is the complement of the three coordinate hyperplanes $z_1 = 0$, $z_2 = 0$ and $z_3 = 0$. The set in $\mathbb{C}^3 - \{0\}$ given by the equation $(\star)$

$$\lambda_1|z_1|^2 + \lambda_2|z_2|^2 + \lambda_3|z_3|^2 = 0 \qquad (\star)$$

meets every leaf in $\mathcal{U}$ in exactly one point. So that the space of leaves in $\mathcal{U}$ can be identified with the set, also denoted by M, satisfying this equation.
The set M is a complex cone with the origin deleted so that if $Z \in M$ also $cZ \in M$ for al $c \in \mathbb{C}^*$.
Hence one has a free action of $\mathbb{C}^*$ and the quotient $N := M/\mathbb{C}^*$, then a complex, compact manifold of dimension one. In fact N is an elliptic curve.

Any elliptic curve is obtained this way.

We see that N is the projectivization of M and therefore N can be identified is the set of points satisfying the following two equations:

$$\lambda_1|z_1|^2 + \lambda_2|z_2|^2 + \lambda_3|z_3|^2 = 0$$

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$$

modulo the natural action of the circle given by

$$(z_1, z_2, z_3) \mapsto (\mu z_1, \mu z_2, \mu z_3), \ |\mu| = 1, \ (z_1, z_2, z_3) \in N.$$

We can generalize this to linear actions of $\mathbb{C}^m$ on $\mathbb{C}^n$ when $n > 2m$.

Let m be a positive integer and n an integer greater than $2m$.

Let $\Lambda = (\Lambda_1, \dots, \Lambda_n)$ be a configuration of n vectors of $\mathbb{C}^m$.
Let $\mathcal{H}(\Lambda_1, \cdots, \Lambda_n) \subset \mathbb{C}^m$ be the convex hull of the n-tuple $\Lambda = (\Lambda_1, \dots, \Lambda_n)$.

One says that Λ ist *admissible* if the following holds true:

1) The *Siegel condition*:
The origin 0 belongs to the convex hull $\mathcal{H}(\Lambda_1, \dots, \Lambda_n)$.

2) The *weak hyperbolicity condition*:
For every $2m$-tuple $(i_1, \cdots, i_{2m})$, $1 \leq i_1 < \cdots < i_{2m} \leq n$ (recall $n > 2m$), one has: $0 \notin \mathcal{H}(\Lambda_{i_1}, \cdots, \Lambda_{i_1})$.

Let $\mathcal{F}$ be the holomorphic foliation of complex projective $(n-1)$-space $\mathbb{P}^{n-1}$ given by the action

$$(T, [z]) \in \mathbb{C}^m \times \mathbb{P}^{n-1} \longmapsto [z_1 \cdot \exp\langle \Lambda_1, T \rangle, \ldots, z_n \cdot \exp\langle \Lambda_n, T \rangle] \in \mathbb{P}^{n-1}$$

$$T = (t_1, \cdots, t_m) \in \mathbb{C}^m, \ \Lambda_i \in \mathbb{C}^m.$$

The brackets denote projective homogeneous coordinates of the corresponding projective space:

$$[z] := [z_1, \cdots, z_n]$$

and $\langle -, - \rangle$ defined by:

$$\langle [z], [w] \rangle := z_1 w_1 + \cdots + z_m w_m$$

is the inner product (*not* the hermitian product).

This action of $\mathbb{C}^m$ is the projectivization of the linear action in $\mathbb{C}^n$ given by the family of m linear commuting vector fields of $\mathbb{C}^n$ given by the diagonal matrices whose eigenvalues are the entries of Λ_i.
Therefore, consider the lifting of this action and the corresponding foliation $\tilde{\mathcal{F}}$ of $\mathbb{C}^n$ given by the orbits of the action:

$$(T,z) \in \mathbb{C}^m \times \mathbb{C}^n \longmapsto (z_1 \cdot \exp\langle \Lambda_1, T\rangle, \dots, z_n \cdot \exp\langle \Lambda_n, T\rangle) \in \mathbb{C}^n$$

If $z \in \mathbb{C}^n$ we say that the leaf (or orbit) $L(z)$ of the action of $\mathbb{C}^m$ is a Siegel leaf if 0 *it is not* in the closure of $L(z)$. If 0 is in the closure of $L(z)$ we say that the leaf $L(z)$ is of Poincaré type.

Next we will describe an open set $\mathcal{S}$, of $\mathbb{C}^n$, where the space of leaves is Hausdorff.

For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, let $I(z) = \{j \in \{1, 2 \cdots, n\} | \; z_j \neq 0\}$

$$\mathcal{S} = \{z \in \mathbb{C}^n \quad | \quad 0 \in \mathcal{H}(\{\Lambda_i | \; i \in I(z)\})\}$$

and let $\mathcal{V}$ be the image of $\mathcal{S}$ in $\mathbb{P}^{n-1}$.
Finally, let

$$\mathcal{T} = \{z \in \mathbb{C}^n \quad | \quad \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0\}$$

and

$$\mathcal{N} = \{[z] \in \mathbb{P}^{n-1} \quad | \quad \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0\}$$

We see from its definition that $\mathcal{S} = \mathbb{C}^n - E$ where E is an analytic set, whose different components correspond to subspaces of $\mathbb{C}^n$ where some coordinates vanish. Therefore $\mathcal{S}$ contains $(\mathbb{C}^*)^n$ and it is invariant under the natural action on $\mathbb{C}^n$ of $(\mathbb{C}^*)^n$ via diagonal and invertible matrices.
Another characterization of $\mathcal{S}$ is the following:

$$\mathcal{S} = \{z \in \mathbb{C}^n \mid 0 \text{ is not in the closure of the leaf of } \tilde{\mathcal{F}} \text{ through } z\}$$

in other words $\mathcal{S}$ is the union of the Siegel Leaves and it open and invariant under the action of $\mathbb{C}^m$

The weak hyperbolicity condition implies that the system of quadratic equations which define $\mathcal{T}$ et $\mathcal{N}$, given before, are of maximal rank in in every point.

The Siegel condition implies that both $\mathcal{T}$ and $\mathcal{N}$ nonempty. One also shows that $\tilde{\mathcal{F}}$ is a non singular foliation when restricted to $\mathcal{S}$ and that $\mathcal{T}$ is a smooth manifold which meets every leaf of $\tilde{\mathcal{F}}$ contained in $\mathcal{S}$ and it is transverse to the leaves. In other words: the quotient space $\tilde{\mathcal{F}}$ restricted to $\mathcal{S}$ can be canonically identified to $\mathcal{T}$ and therefore it is a Hausdorff space.

An important fact is that $\mathcal{T}$ can be given the structure of a complex manifold which we denote by M.

In the same way, $\mathcal{N}$ can be identified with $\mathcal{F}$ restricted to V and therefore becomes a compact complex manifold. Let us denote by N this complex manifold. The complex dimension of M is $n - m$ and the complex dimension of N is $n - m - 1$.

The natural projection $M \to N$, induced by the projection $\mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$, is a principal $\mathbb{C}^*$-bundle.
Let M_1 denote the total space of the associated circle bundle. It has the same homotopy type as M but has the advantage of being compact. Let us note that M_1 can be identified with the transverse intersection of the cone $\mathcal{T}$ with the unit sphere $\mathbb{S}^{2n-1}$ of $\mathbb{C}^n$.
Therefore we define:

$$M_1 = \{z \in \mathbb{C}^n \quad | \quad \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0, \ \sum_{i=1}^{n} |z_i|^2 = 1\}$$

The space $\mathcal{S}$ has the same homotopy of M and therefore the same homotopy type of M_1.

(i) If $n = 2m+1$, then the convex hull of the Λ_i's is combinatorially equal to the $2m+1$-simplex of $\mathbb{C}^m \simeq \mathbb{R}^{2m}$. If we take out one of the Λ_i's, 0 does not belong to the convex closure of the others. In other words, $\mathcal{S}$ is equal to $(\mathbb{C}^*)^n$ One can show that N is a complex torus and that *every complex torus is obtained this way*.

(ii) If $m = 1$ Let us define for, $n \geq 4$:

$$\Lambda_1 = 1 \qquad \Lambda_2 = i \qquad \Lambda_3 = \ldots = \Lambda_n = -1 - i\ .$$

One can prove in this case that $\mathcal{S}$ is equal to $(\mathbb{C}^*)^2 \times \mathbb{C}^{n-2} \setminus \{0\}$.
Let us consider the two equations which define $\mathcal{T}$:

$$|z_1|^2 = |z_3|^2 + \ldots + |z_n|^2$$

$$|z_2|^2 = |z_3|^2 + \ldots + |z_n|^2.$$

If we intersect $\mathcal{T}$ with the unit sphere $\mathbb{C}^n$ we see that this intersection is diffeomorphic to $\mathbb{S}^{2n-5} \times \mathbb{S}^1 \times \mathbb{S}^1$ and one shows that N is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{2n-5}$. In particular, for $n = 4$, one obtains all linear Hopf surfaces.

(iii) If $m = 1$ let:

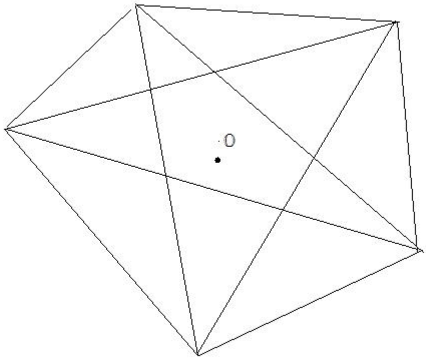
$$\Lambda_1 = 1 \qquad \Lambda_2 = \Lambda_3 = i \qquad \Lambda_4 = \Lambda_5 = -1 - i \ .$$

The same reasoning as before shows that N is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^3$.

On obtains this in this way examples of Calabi-Eckmann.

A Calabi-Eckmann manifold is a complex manifold whose underlying smooth manifold is a product of odd-dimensional spheres $\mathbb{S}^{2n-1} \times \mathbb{S}^{2m-1}$.
One can show that every linear Calabi-Eckmann manifold is obtained this way.

(iv) In the case that one has the configuration given by the vertices of the pentagon



Santiago López de Medrano has shown that M_1 is diffeomorphic to the connected sum of five copies of $\mathbb{S}^3 \times \mathbb{S}^4$. The manifold N, is the quotient of M_1 by the orbits of a free action of $\mathbb{S}^1$.

In the examples (ii) and (iv), one obtains *non-symplectic* manifolds, since their second de Rham cohomology group vanishes.
This is a general fact in the manifolds we have obtained:
In general the manifold N is a compact,complex manifold which is not symplectic.

THEOREM

The following properties are two-by-two equivalent: (i) N est symplectic.

(ii) N is a Kähler manifold.
(iii) N is a complex torus
(iv) One has $n = 2m + 1$.

Let $(\Lambda_1, \cdots, \Lambda_n)$ be a configuration admissible i.e. it satisfies both the Siegel and weak hyperbolicity conditions as before.
Consider the system of equations:

$$\sum_{i=1}^{n} s_i \Lambda_i = 0$$

$$\sum_{i=1}^{n} s_i = 0$$

We say that the configuration satisfies condition (K) if the dimension over $\mathbb{Q}$ of the vector space of rational solutions of the system above is maximal, in other words is of dimension $n - 2m - 1$.

THEOREM

Let N be one of our manifolds corresponding to a configuration which satisfies condition (K).
Then N is a Seifert fibration in complex torii of dimension m over a quasi-regular, projective, toric variety of dimension $n-2m-1$.

This theorem has the following:

COROLLARY

Let N satisfy the conditions of the above theorem. Then the algebraic reduction of N is a quasi-regular, projective, toric variety of dimension $n-2m-1$.

As a particular case of the previous theorem one recovers the elliptic fibrations used by E. Calabi et B. Eckmann to provide the product of spheres $\mathbb{S}^{2p-1} \times \mathbb{S}^{2q-1}$ (pour $p > 1$ et $q > 1$) with a complex structure. This generalization is given by the following

Definition

A generalized Calabi-Eckmann fibration is the fibration obtained by the previous theorem.

Since we know, fixing m and n, that the set of configurations satisfying condition (K) is dense in the space of admissible configurations on obtains:

COROLLARY

Every manifold N corresponding to an admissible configuration is a small deformation of a generalized Calabi-Eckmann fibration

THEOREM

Let X be a projective, quasi-regular, toric variety. Then there exists $m > 0$ and a manifold N corresponding to an admissible configuration which admits a generalized Calabi-Eckmann over X and whose fibres are complex torii of complex dimension m.

Furthermore, if X is nonsingular (smooth), one can choose m and N such that the fibration is a holomorphic principal fibration .

Let N be as before. Let, as before,

$$M_1 = \{z \in \mathbb{C}^n \quad | \quad \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0, \ \sum_{i=1}^{n} |z_i|^2 = 1\}$$

Let us remark that the standard action of the torus $(\mathbb{S}^1)^n$ on $\mathbb{C}^n$

$$(\exp i\theta, z) \in (\mathbb{S}^1)^n \times \mathbb{C}^n \longmapsto (\exp i\theta_1 \cdot z_1, \ldots, \exp i\theta_n \cdot z_n) \in \mathbb{C}^n \quad (\star\star)$$

leaves M_1 invariant. The quotient of M_1 by this action can be identified, via the difféomorphism $r \in \mathbb{R}^+ \to r^2 \in \mathbb{R}^+$, to

$$K = \{r \in (\mathbb{R}^+)^n \quad | \quad \sum_{i=1}^{n} r_i \Lambda_i = 0, \ \sum_{i=1}^{n} r_i = 1\}$$

LEMMA

The quotient K is a convex polytope of dimension $n - 2m - 1$ with $n - k$ hyperfaces.

Proof. By definition K is the intersection of the space A of solutions of an affine system with the closed sets $r_i \geq 0$. Each one of these closed sets defines an affine half-space $A \cap \{r_i \geq 0\}$ in the affine space A. In other words, K is the intersection of a finite number of affine half-spaces. Since this intersection is bounded (since M_1 is compact), one obtains indeed a convex polytope. The weak hyperbolicity condition implies that the affine system that defines K is of maximal rank. Hence, K is of dimension $n - 2m - 1$.

Let us consider in more detail the definition of K. The points $r \in K$ verifying $r_i > 0$ for all i are the points which belong to the interior of the convex polytope. They correspond to the points z de M_1 which also belong to $(\mathbb{C}^*)^n$, i.e. to the points of M_1 such that the orbit under the action $(\star\star)$ is isomorphic to $(\mathbb{S}^1)^n$. The points which belong to a hyperface are exactly the points r of K having all of its coordinates *except one* equal to zero. They correspond to the points z de M_1 which have a unique coordinate equal to zero, i.e. such that its orbit under the action $(\star\star)$ is isomorphic to $(\mathbb{S}^1)^{n-1}$. One obtains from the definition of K that there exist points of K having all coordinates different from zero except the ith coordinate if and only 0 belongs to the convex envelope of the configuration formed by the Λ_j with j different from i; hence if and only if Λ_i is a point which can be eliminated keeping the conditions of Siegel and weak hyperbolicity. therefore one has $n - k$ hyperfaces. $\square$

One calls the convex polytope K *the associated polytope* . One central idea is that the topology of the manifolds M_1, and therefore of the manifolds N, is codified by the combinatorial type of the polytope K. To make this idea more precise, it is interesting to push to the end the reasoning involved in the proof of the preceding lemma. One had seen that

$$K_i = K \cap \{r_i = 0,\ r_j > 0 \text{ for } j \neq i\}$$

is nonempty, and therefore is a hyperface de K, if and only if

$$0 \in \mathcal{H}((\Lambda_j)_{j \neq i}) \ .$$

Analogously, given I a subset of $\{1, \ldots, n\}$, the set

$$K_I = K \cap \{r_i = 0 \text{ for } i \in I,\ r_j > 0 \text{ for } j \notin I\}$$

is nonempty, and therefore it is a facet of K of codimension equal the cardinality of I, if and only if

$$0 \in \mathcal{H}((\Lambda_j)_{j \notin I})$$

One has therefore stablished a very important correspondence between two convex polytopes: the polytope K on one hand and the convex hull of the Λ_i's on the other hand.
This correspondence allows us to to prove the following result:

THEOREM
(i) The polytope K is simple, in other words, it is the dual of a simplicial polytope.
(ii) Let P be a simple convex polytope. Then there exists manifolds N, as described before, whose associated polytope is combinatorially equivalent to P.

Sketch of the proof. The first part is a direct consequence of the existence of the correspondence. One translates the weak hyperbolicity condition in the combinatorics of K to deduce that each vertex of K is a vertex of exactly $n-2m-1$ edges, and this number is precisely the dimension of K. This property characterizes simple polytopes. To prove (ii), one needs to reconstruct the convex hull of the Λ_i's from the polytope P. The correspondence described before can be expressed in the following way: The convex hull of the Λ_i's must be a Gale diagram of the polytope which is dual to P . There are classical methods in combinatorics and convex geometry to construct such diagrams and this permits to finish the proof. □

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