From A_n (Calogero) to H_4 (rational)

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Let us consider the Hamiltonian = the Schrödinger operator

$$\mathcal{H} \;=\; -\Delta \;+\; V(x)\;, \qquad x \in R^d$$

A problem of quantum mechanics is to solve the Schrödinger equation

$$\mathcal{H}\Psi(x) \;=\; E\Psi(x) \quad , \quad \Psi(x) \in L^2(R^d)$$

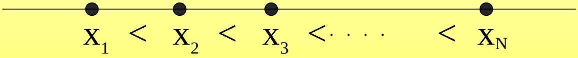
finding the spectra (the energies and eigenfunctions).
The Hamiltonian is an infinite-dimensional matrix

To solve the Schrödinger equation ⇒ diagonalize the infinite-dimensional matrix

It is transcendental problem, the characteristic polynomial is of infinite order and it has infinitely-many roots. *Do exist cases when roots (energies) can be found explicitly (exactly)?*

- Calogero Model (A_{N-1} Rational model)
  (F. Calogero, '69)

  N identical particles on a line with singular pairwise interaction



$$\mathcal{H}_{\mathrm{Cal}} = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial {x_i}^2} + \omega^2 {x_i}^2\right) + \nu(\nu-1)\sum_{i>j}^{N}\frac{1}{(x_i - x_j)^2}$$

Symmetry: S_n (permutations $x_i \to x_j$) and $\mathbb{Z}_2$ (all $x_i \to -x_i$)

$$\Psi_0(x) = \prod_{i<j} |x_i - x_j|^\nu e^{-\frac{\omega}{2}\sum x_i^2} \ ,$$

$$h_{\rm Cal} = 2\Psi_0^{-1}\left(\mathcal{H}_{\rm Cal} - E_0\right)\Psi_0$$

$$Y = \sum x_i \ , \ \ y_i = x_i - \frac{1}{N} Y \ , \ \ i = 1, \ldots, N$$

$$(x_1, x_2, \ldots x_N) \to \big(Y, t_n(x) = \sigma_n(y(x))|\ n = 2, 3 \ldots N\big)$$

$$\sigma_k(x) = \sum_{i_1<i_2<\ldots<i_k} x_{i_1} x_{i_2} \ldots x_{i_k}$$

$$\sigma_k(-x) = (-)^k \sigma_k(x)$$

$$t_1 = 0 \ , \ \ t_2 = \sum_{i<j} (x_i - x_j)^2 = r^2$$

radius in space of relative coordinates

After separation cms,

$$h_{\mathrm{Cal}} = \mathcal{A}_{ij}(t)\frac{\partial^2}{\partial t_i \partial t_j} + \mathcal{B}_i(t)\frac{\partial}{\partial t_i}$$

$$\mathcal{A}_{ij} = \frac{(N-i+1)(1-j)}{N}\, t_{i-1}\, t_{j-1} + \sum_{l \geq \max(1,j-i)} (2l-j+i) t_{i+l-1} t_{j-l-1}$$

$$\mathcal{B}_i = \frac{1}{N}(1+\nu N)(N-i+2)(N-i+1)\, t_{i-2} + 2\omega\,(i-1)\, t_i$$

★ Eigenvalues:

$$\epsilon_{\{p\}} \;=\; 2\omega \sum_{i=2}^{N} (i-1)\; p_i$$

the spectra linear in quantum numbers and of *anisotropic* harmonic oscillator with ratios $\quad 1 : 2 : 3 : ... : (N-1)$

★ Hamiltonian h has infinitely many finite-dimensional invariant subspaces

$$\mathcal{P}_n^{(N-1)} \;=\; \langle {t_2}^{p_2} {t_3}^{p_3} \ldots {t_N}^{p_N} |\; 0 \le \Sigma p_i \le n \rangle$$

where $n = 0, 1, 2, \ldots$

gl_{d+1}-algebra (acting in R^d)

(almost degenerate or totally symmetric, Young tableaux is a row)

$$(n, \underbrace{0, 0, \ldots 0}_{d-1})$$

$$
\begin{aligned}
\mathcal{J}_i^- &= \frac{\partial}{\partial t_i}, \qquad i = 1, 2 \ldots d\ , \\
\mathcal{J}_{ij}{}^0 &= t_i \frac{\partial}{\partial t_j}, \qquad i, j = 1, 2 \ldots d\ , \\
\mathcal{J}^0 &= \sum_{i=1}^{d} t_i \frac{\partial}{\partial t_i} - n\,, \\
\mathcal{J}_i^+ &= t_i \mathcal{J}^0 = t_i \left(\sum_{j=1}^{d} t_j \frac{\partial}{\partial t_j} - n \right), \quad i = 1, 2 \ldots d\ .
\end{aligned}
$$

- $(d+1)^2$ generators

- if $n = 0, 1, 2 \ldots$, *fin-dim* irreps

$$\mathcal{P}_n^{(d)} = \langle t_1^{p_1} t_2^{p_2} \ldots t_d^{p_d} \,|\, 0 \le \Sigma p_i \le n \rangle$$

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_n \subset \ldots \mathcal{P} \ ,$$

then such a construction is called *infinite flag (filtration)* $\mathcal{P}$.
Remark. The flag $\mathcal{P}^{(d)}$ is made out of finite-dimensional irreducible representation spaces $\mathcal{P}_n^{(d)}$ of the algebra gl_{d+1} taken in realization $(*)$.

Any operator made out of generators $(*)$ has finite-dimensional invariant subspace which is finite-dimensional irreducible representation space and visa versa.

★ Hamiltonian:

$$h_{\rm Cal} \;=\; Pol_2(\mathcal{J}_i^- \;,\; \mathcal{J}_{ij}{}^0)$$

$gl(N-1)$ is the hidden algebra of N-body Calogero model.
It is $gl(N-1)$ quantum top in constant magnetic field.

★ Eigenfunctions:

they are elements of the flag of polynomials $\mathcal{P}^{(N-1)}$.
Each subspace $\mathcal{P}_n^{(N-1)}$ contains C_{n+N-1}^{N-1} eigenfunctions
(volume of the Newton polytope (prism))

Remark:

Calogero Hamiltonian $\mathcal{H}_{\rm Cal}$ has 2nd order integral.

After separation cms, the relative Hamiltonian admits separation of radial variable

$$\mathcal{H}^{(rel)}_{\rm Cal} \;=\; -\frac{1}{2r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right)+\omega^2 r^2+\frac{1}{2r^2}\underbrace{(-\Delta^{(N-1)}_{\Omega}+\mathcal{W}(\Omega))}_{\mathcal{F}_{\rm Cal}}$$

where $\{\Omega\}$ are coordinates on $S^{(N-1)}$-sphere.

Evidently, the commutator

$$[\mathcal{H}_{\rm Cal}\;,\;\mathcal{F}_{\rm Cal}] \;=\; 0$$

Gauge-rotated integral

$$f_{\mathrm{Cal}} \;=\; \Psi_0^{-1}\left(\mathcal{F}_{\mathrm{Cal}} - F_0\right)\Psi_0$$

where $\mathcal{F}_{\mathrm{Cal}}\Psi_0 = F_0\Psi_0$, is algebraic,

$$f_{\mathrm{Cal}} \;=\; f_{ij}(t)\frac{\partial^2}{\partial t_i \partial t_j} + g_i(t)\frac{\partial}{\partial t_i}$$

where f_{ij} is 2nd degree polynomial, $f_{2j} = 0$
g_i is 1st degree polynomial, $g_2 = 0$

$$f_{\mathrm{Cal}} \;=\; Pol_2(\mathcal{J}_i^- \;,\; \mathcal{J}_{ij}{}^0)$$

in $gl(N-1)$ generators

$$[h_{\mathrm{Cal}}(\mathcal{J}), f_{\mathrm{Cal}}(\mathcal{J})] \;\neq\; 0$$

$sl(2)$-*Quasi-Exactly-Solvable generalization*

By adding to $h_{\rm Cal}$, the operator

$$\delta h^{(qes)} = 4(at_2^2 - \gamma)\frac{\partial}{\partial t_2} - 4akt_2 + 2\omega k$$

we get $h_{\rm Cal} + \delta h^{(qes)}$ having fin-dim invariant subspace

$$\mathcal{P}_k = \langle t_2^p | 0 \leq p \leq k \rangle$$

Making a gauge rotation of $h_{\rm Cal} + \delta h^{(qes)}$
and change of variables to Cartesian the Hamiltonian becomes

$$\mathcal{H}^{(2)}_{Cal} = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial {x_i}^2} + \omega^2 x_i^2\right) + \nu(\nu-1)\sum_{j<i}^{N}\frac{1}{(x_i-x_j)^2} +$$

$$\frac{2\gamma\left[\gamma - 2n(1+\nu+\nu n)+3\right]}{r^2} +$$

$$+\ a^2 r^6\ +\ 2a\omega r^4\ -\ a\left[2k + 2n(1+\nu+\nu n) - \gamma - 1\right] r^2,$$

for which $(k+1)$ eigenfunctions are of the form

$$\Psi^{(\mathrm{qes})}_k(x) = \prod_{i<j}^{n} |x_i - x_j|^{\nu} (r^2)^{\gamma} P_k(r^2)\ \exp\left[-\frac{\omega}{2}\sum_{k=1}^{n} x_i^2 - \frac{a}{4} r^4\right]\ ,$$

where P_k is a polynomial of degree k in $r^2 = \sum_{i<j}(x_i - x_j)^2$

Hamiltonian Reduction Method

(Olshanetsky-Perelomov '77, Kazhdan-Kostant-Sternberg '78)

- Define Laplace-Beltrami operators on symmetric spaces of simple Lie groups (free/harmonic oscillator motion)
- Radial parts of L-B operators $\equiv$ Olshanetsky-Perelomov Hamiltonians relevant from physical point of view. They can be associated with root systems.

Rational case:

$$\mathcal{H} = \frac{1}{2}\sum_{k=1}^{N}\left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2\right] + \frac{1}{2}\sum_{\alpha\in R_+}\nu_{|\alpha|}(\nu_{|\alpha|}-1)\frac{|\alpha|^2}{(\alpha\cdot x)^2}$$

where R_+ is a set of positive roots and $\nu_{|\alpha|}$ are coupling constants depending on the root length.

For all roots of the same length $\nu_{|\alpha|} = \nu$.

★ They take discrete values but can be generalized to any value.

Configuration space - Weyl chamber.

Ground state wave function

$$\Psi_0(y) \;=\; \prod_{\alpha \in R_+} |(\alpha \cdot y)|^{\nu_{|\alpha|}}\, e^{-\omega y^2/2}$$

The Hamiltonian is completely-integrable (super-integrable) and exactly-solvable for **any** value of $\nu > -\frac{1}{2}$ and $\omega > 0$. It is invariant wrt Weyl (Coxeter) group transformation (symmetry group of root space)

Procedure:

- Gauging away ground state eigenfunction (*similarity transformation*) $(\Psi_0)^{-1}(\mathcal{H} - E_0)\Psi_0 \ = \ h$

- Olshanetsky-Perelomov Hamiltonians (OPH) possess different symmetries (permutations, translation-invariance, reflections, periodicity etc). These symmetries correspond to the Weyl (Coxeter) group plus translations. By coding these symmetries to new coordinates (*taking the Weyl (Coxeter) invariants as new coordinates*) we find 'premature' (undressed by symmetries) operators to these Hamiltonians.

Example: $Weyl(A_n) \ = \ S_n + T$

WHAT ARE THESE COORDINATES?

- Weyl (Coxeter) polynomial invariants:

$$t_a^{(\Omega)}(x) = \sum_{\alpha \in \Omega} (\alpha, x)^a \ ,$$

where a's are the *degrees* of the Weyl (Coxeter) group W and Ω is an orbit.
The invariants t are defined ambiguously, up to invariants of lower degrees, they depend on chosen orbit.
but **always** lead to rational OPH h in a form of algebraic operator with polynomial coeffs.

- BC_N –Rational model

$$\mathcal{H}_{BC_N} = -\frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial^2}{\partial {x_i}^2} - \omega^2 x_i^2\right)$$

$$+\nu(\nu-1)\sum_{i<j}\left[\frac{1}{(x_i-x_j)^2}+\frac{1}{(x_i+x_j)^2}\right]+\ \frac{\nu_2(\nu_2-1)}{2}\sum_{i=1}^{N}\frac{1}{x_i^2}$$

Symmetry: $S_n \oplus (\mathbb{Z}_2)^{\otimes n}$ (permutations $x_i \to x_j$ and $x_i \to -x_i$)

$$\Psi_0 = \left[\prod_{i<j} |x_i - x_j|^\nu |x_i + x_j|^\nu \prod_{i=1}^{N} |x_i|^{\nu_2}\right] e^{-\frac{\omega}{2}\sum_{i=1}^{N} x_i^2} \ ,$$

$$h_{BC_N} = (\Psi_0)^{-1} \left(\mathcal{H}_{BC_N} - E_0\right) \Psi_0$$

$$(x_1, x_2, \ldots x_N) \rightarrow \left(\sigma_k(x^2)\big|_{\ k=1,2,\ldots,N}\right)$$

$$\sigma_k(x^2) = \sum_{i_1<i_2<\cdots<i_k} x_{i_1}^2 x_{i_2}^2 \cdots x_{i_k}^2$$

$$\sigma_1(x^2) = x_1^2 + x_2^2 + \ldots + x_N^2 \ = \ r^2$$

$$h_{BC_N} = \mathcal{A}_{ij}(\sigma)\frac{\partial^2}{\partial\sigma_i\partial\sigma_j} + \mathcal{B}_i(\sigma)\frac{\partial}{\partial\sigma_i}$$

$$\begin{aligned}
\mathcal{A}_{ij} &= -2\sum_{l\geq 0}(2l+1+j-i)\,\sigma_{i-l-1}\,\sigma_{j+l} \\
\mathcal{B}_i &= [1+\nu_2+2\nu(N-i)]\,(N-i+1)\,\sigma_{i-1} + 2\,\omega\,i\,\sigma_i
\end{aligned}$$

★ Eigenvalues:

$$\epsilon_n \;=\; 2\omega \sum_{i=1}^{N} i\; n_i$$

the spectra linear in quantum numbers and of *anisotropic* harmonic oscillator with ratios $1:2:3:...:N$

★ Hamiltonian h has infinitely many finite-dimensional invariant subspaces

$$\mathcal{P}_n^{(N)} \;=\; \langle {\sigma_1}^{p_1} {\sigma_2}^{p_2} \ldots {\sigma_N}^{p_N} |\; 0 \leq \Sigma p_i \leq n \rangle$$

where $n = 0, 1, 2, \ldots$

★ Hamiltonian:

$$h_{\mathrm{BC_N}} \;=\; Pol_2(\mathcal{J}_i^{-}\;,\;\mathcal{J}_{ij}{}^{0})$$

$gl(N)$ is the hidden algebra of BC_N rational model.
It is $gl(N)$ quantum top in constant magnetic field.

★ Eigenfunctions:

they are elements of the flag of polynomials $\mathcal{P}^{(N)}$.
Each subspace $\mathcal{P}_n^{(N)}$ contains C_{n+N}^{N} eigenfunctions
(volume of the Newton polytope (prism))

Remark:

BC_N Hamiltonian admits 2nd order integral as result of separation of radial variable

$$\mathcal{H}_{BC_N} \;=\; -\frac{1}{2r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right)+\omega^2 r^2+\frac{1}{2r^2}\underbrace{(-\Delta_{\Omega}^{(N-1)}+\mathcal{W}(\Omega))}_{\mathcal{F}_{BC_N}}$$

where $\{\Omega\}$ are coordinates on $S^{(N)}$-sphere.

Evidently, the commutator

$$[\mathcal{H}_{BC_N}\ ,\ \mathcal{F}_{BC_N}] \;=\; 0$$

Gauge-rotated integral

$$f_{BC_N} \;=\; \Psi_0^{-1}\,(\mathcal{F}_{BC_N} - F_0)\,\Psi_0$$

where $\mathcal{F}_{BC_N}\Psi_0 = F_0\Psi_0$, is algebraic,

$$f_{BC_N} \;=\; f_{ij}(t)\frac{\partial^2}{\partial t_i \partial t_j} + g_i(t)\frac{\partial}{\partial t_i}$$

where f_{ij} is 2nd degree polynomial, $f_{1j} = 0$
g_i is 1st degree polynomial, $g_1 = 0$

$$f_{BC_N} \;=\; Pol_2(\mathcal{J}_i^- \;,\; \mathcal{J}_{ij}{}^0)$$

in $gl(N)$ generators

$$[h_{BC_N}(\mathcal{J}), f_{BC_N}(\mathcal{J})] \;\neq\; 0$$

sl(2)-*Quasi-Exactly-Solvable generalization*

By adding to h_{BC_N}, the operator (the same as for Calogero model)

$$\delta h^{(qes)} = 4(a\sigma_1^2 - \gamma)\frac{\partial}{\partial\sigma_1} - 4ak\sigma_1 + 2\omega k$$

we get $h_{BC_N} + \delta h^{(qes)}$ having fin-dim invariant subspace

$$\mathcal{P}_k = \langle \sigma_1^p | 0 \le p \le k \rangle$$

Making a gauge rotation of $h_{BC_N} + \delta h^{(qes)}$
and change of variables to Cartesian the Hamiltonian becomes

$$\mathcal{H}^{(1)}_{BC_N} = -\frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial^2}{\partial {x_i}^2} - \omega^2 x_i^2\right)$$

$$+\nu(\nu-1)\sum_{i<j}\left[\frac{1}{(x_i-x_j)^2}+\frac{1}{(x_i+x_j)^2}\right] + \frac{\nu_2(\nu_2-1)}{2}\sum_{i=1}^{N}\frac{1}{x_i^2} +$$

$$\frac{2\gamma\left[\gamma - 2N(1+2\nu(N-1)+\nu_2)+3\right]}{r^2} +$$

$$+\ a^2 r^6 \ +\ 2a\omega r^4 \ -\ a\left[2k + 2N(1+2\nu(N-1)+\nu_2) - \gamma - 1\right] r^2,$$

for which $(k+1)$ eigenfunctions are of the form

$$\Psi_k^{(\mathrm{qes})}(x) = \prod_{i<j}^{n} |x_i^2 - x_j^2|^{\nu} \prod_{i=1}^{n} |x_i|^{\nu_2} (r^2)^{\gamma} P_k(r^2)\ \exp\left[-\frac{\omega r^2}{2} - \frac{a}{4} r^4\right],$$

- Both A_N- and BC_N- rational (and trigonometric) models possess **algebraic** forms associated with preservation of the **same** flag of polynomials $\mathcal{P}^{(N)}$. The flag is invariant wrt linear transformations in space of orbits $t \mapsto t + A$. It preserves the algebraic form of Hamiltonian.

- Their Hamiltonians (as well as higher integrals) can be written in the Lie-algebraic form

$$h = P_2(\mathcal{J}(b \subset gl^{(*)}_{N+1}))$$

  where P_2 is a polynomial of 2nd degree in generators $\mathcal{J}$ of the maximal affine subalgebra of the algebra gl_{N+1} in realization $(*)$. Hence gl_{N+1} is their **hidden algebra**. From this viewpoint all four models are different faces of a **single** model.

- *Supersymmetric A_N- and BC_N- rational and trigonometric models possess* **algebraic** *forms, preserve the* **same** *flag of (super)polynomials and their* **hidden algebra** *is the superalgebra $gl(N+1|N)$.*

In the connection to flags of polynomials we introduce a notion *'characteristic vector'*.

Let us consider a flag made out of "triangular" linear space of polynomials

$$\mathcal{P}^{(d)}_{n,\vec{f}} \;=\; \langle x_1^{p_1} x_2^{p_2} \ldots x_d^{p_d} | 0 \leq f_1 p_1 + f_2 p_2 + \ldots + f_d p_d \leq n \rangle$$

where the "grades" f's are positive integer numbers and $n = 0, 1, 2, \ldots$.

DEFINITION. Characteristic vector is a vector with components α_i :

$$\vec{f} = (f_1, f_2, \ldots f_d).$$

The characteristic vector for flag $\mathcal{P}^{(d)}$:

$$\vec{f}_0 \;=\; \underbrace{(1, 1, \ldots 1)}_{d}$$

Wolves model (G_2 – Rational model) (Wolves, '75)

$$\mathcal{H}_{G_2} = \frac{1}{2}\sum_{i=1}^{3}\left(-\frac{\partial^2}{\partial {x_i}^2} + \omega^2 x_i^2\right) + \nu(\nu-1)\sum_{i<j}^{3}\frac{1}{(x_i - x_j)^2} + 3\mu(\mu-1)\sum_{k<l,\quad k,l\neq m}^{3}\frac{1}{(x_k + x_l - 2x_m)^2}$$

Symmetry: dihedral group D_6

$$\Psi_0 = \prod_{i<j}^{3}|x_i - x_j|^{\nu}\prod_{k<l,\ k,l\neq m}^{3}|x_i + x_j - 2x_k|^{\mu} e^{-\frac{1}{2}\omega\sum x_i^2}$$

$$h_{G_2} = (\Psi_0)^{-1}\left(\mathcal{H}_{G_2} - E\right)\Psi_0$$

$$Y = \sum x_i\ ,\ y_i = x_i - \frac{1}{3}Y\ ,\ i = 1, 2, 3$$

$$\lambda_1 = -y_1^2 - y_2^2 - y_1 y_2 = r^2 \quad , \quad \lambda_2 = [y_1 y_2 (y_1 + y_2)]^2$$

After cms separation

$$h_{\mathrm{G}_2} = \lambda_1 \partial^2_{\lambda_1 \lambda_1} + 6\lambda_2 \partial^2_{\lambda_1 \lambda_2} - \frac{4}{3}\lambda_1^2 \lambda_2 \partial^2_{\lambda_2 \lambda_2}$$

$$+\{2\omega\lambda_1 + 2[1 + 3(\mu + \nu)]\}\partial_{\lambda_1} + \left[6\omega\lambda_2 - \frac{4}{3}(1 + 2\mu)\lambda_1^2\right]\partial_{\lambda_2}$$

★ Eigenvalues:

$$\epsilon_{\{p\}} \; = \; 2\omega(p_1 + 3p_2)$$

the spectra of *anisotropic* harmonic oscillator with frequency ratio 1 : 3

★ Hamiltonian h has infinitely many finite-dimensional invariant subspaces

$$\mathcal{P}^{(2)}_{n,(1,2)} \; = \; \langle {\lambda_1}^{p_1} {\lambda_2}^{p_2} | \; 0 \le p_1 + 2p_2 \le n \rangle \; , \; n = 0, 1, 2, \ldots$$

The flag $\mathcal{P}^{(2)}_{(1,2)}$ with $\vec{f} = (1,2)$ is preserved by h_{G_2}

★ Eigenfunctions:

they are elements of the flag of polynomials $\mathcal{P}^{(2)}_{(1,2)}$.

Each subspace $\mathcal{P}^{(2)}_{n,(1,2)}$ contains # eigenfunctions equals to volume of the Newton polygone

What about hidden algebra? Does exist algebra for which $\mathcal{P}^{(2)}_{n,(1,2)}$ is the space of (irreducible) representation?

The Lie algebra:

$$J^1 = \partial_t$$
$$J^2_n = t\partial_t - \frac{n}{3}\ ,\ J^3_n = 2u\partial_u - \frac{n}{3}$$
$$J^4_n = t^2\partial_t + 2tu\partial_u - nt$$

$$R_i = t^i\partial_u\ ,\ i = 0, 1, 2\ , \quad \mathcal{R}^{(2)} \equiv (R_0, R_1, R_2)$$

they span non-semi-simple algebra $gl(2, \mathbf{R}) \ltimes \mathcal{R}^{(2)}$

S. Lie, $\sim$1890 at $n = 0$ and A. González-Lopéz et al, '91 at $n \neq 0$ (Case 24)

$$\mathcal{P}^{(2)}_n = (t^p u^q | 0 \leq (p + 2q) \leq n)$$

common invariant subspace (reducible)

By adding

$$T_0^{(2)} \; = \; u\partial_t^2$$

to $gl(2,\mathbf{R}) \ltimes \mathcal{R}^{(2)}$, the action on $\mathcal{P}_n^{(2)}$ gets irreducible.

Property:

$$T_i^{(2)} = \underbrace{[J^4,[J^4,[\ldots J^4, T_0^{(2)}]\ldots]}_{i} = u\partial_t^{2-i} J_0(J_0+1)\ldots(J_0+i-1)\ ,$$

$i = 1, 2, 3$, all of the fixed degree 2, $\quad J_0 = t\partial_t \; + \; 2u\partial_u \; - \; n$

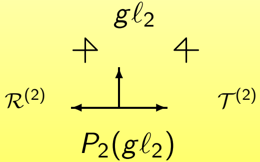
Nilpotency:

$$T_i^{(2)} = 0\ ,\ \ i > 2\ .$$

Commutativity:

$$[T_i^{(2)}, T_j^{(2)}] = 0 \ , \quad i, j = 0, 1, 2 \ , \quad \mathcal{T}^{(2)} \equiv (T_0^{(2)}, T_1^{(2)}, T_2^{(2)})$$

Decomposition: $\qquad g^{(2)} \doteq \mathcal{T}^{(2)} \rtimes gl_2 \ltimes \mathcal{R}^{(2)}$



Infinite-dimensional, 10-generated algebra with $\mathcal{P}_n^{(2)}$ irreps space (seven generators of 1st order and three of 2nd)

$$h_{G_2} = (J^2 + 3J^3)J^1 - \frac{2}{3}J^3 R_2 + 2[3(\mu + \nu) + 1]J^1$$
$$+ 2\omega J^2 + 3\omega J^3 - \frac{4}{3}(1 + 2\mu)R_2$$

Hence, $gl(2, \mathbf{R}) \ltimes \mathcal{R}^{(2)}$ is hidden algebra

★

(i) G_2 Hamiltonian admits two mutually-non-commuting integrals: of 2nd order integral as result of separation of radial variable r^2 and of the 6th order.

(ii) Both integrals after gauge rotation with Ψ_0 take in variables $\lambda_{1,2}$ the algebraic form. Both preserve the same flag $\mathcal{P}^{(2)}_{(1,2)}$.

(iii) Both integrals can be rewritten in term of generators of the algebra $g^{(2)}$: integral of 2nd order in terms of $gl(2, \mathbf{R}) \ltimes \mathcal{R}^{(2)}$ only and of the 6th order contains generators from $\mathcal{T}^{(2)}$ as well.

★
$sl(2)$-Quasi-Exactly-Solvable generalization

By adding to h_{G_2}, the operator (the same as for Calogero model and BC_N)

$$\delta h^{(qes)} = 4(a\lambda_1^2 - \gamma)\frac{\partial}{\partial\lambda_1} - 4ak\lambda_1 + 2\omega k$$

we get $h_{G_2} + \delta h^{(qes)}$ having fin-dim invariant subspace

$$\mathcal{P}_k = \langle \lambda_1^p | 0 \leq p \leq k \rangle$$

Making a gauge rotation of $h_{G_2} + \delta h^{(qes)}$
and change of variables $(Y, \lambda_{1,2})$ to Cartesian the Hamiltonian becomes

$$\mathcal{H}^{(1)}_{G_2} = -\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial^2}{\partial {x_i}^2} - \omega^2 x_i^2\right) +$$

$$\nu(\nu-1)\sum_{i<j}^{3}\frac{1}{(x_i-x_j)^2} + 3\mu(\mu-1)\sum_{i<l,\ i,l\neq m}^{3}\frac{1}{(x_i+x_l-2x_m)^2} +$$

$$\frac{4\gamma(\gamma+3\mu+3\nu)}{r^2} +$$

$$a^2 r^6 \ + \ 2a\omega r^4 \ + \ 2a\left[2k - 3(\mu+\nu) - 2(\gamma+1)\right] r^2,$$

for which $(k+1)$ eigenfunctions are of the form

$$\Psi_k^{(\mathrm{qes})} =$$

$$\prod_{i<j}^{3}|x_i - x_j|^{\nu} \prod_{i<j;i,j\neq p}^{3}|x_i + x_j - 2x_p|^{\mu}(r^2)^{\gamma} P_k(r^2)\exp\left[-\frac{\omega}{2}\sum_{i=1}^{3}x_i^2 - \frac{a}{4}r^4\right]$$

The H_3 rational Hamiltonian

$$\mathcal{H}_{H_3} = \frac{1}{2}\sum_{k=1}^{3}\left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 + \frac{\nu(\nu-1)}{x_k^2}\right] + 2\nu(\nu-1)\sum_{\{i,j,k\}}\sum_{\mu_{1,2}=0,1}\frac{1}{[x_i + (-1)^{\mu_1}\varphi_+ x_j + (-1)^{\mu_2}\varphi_- x_k]^2}$$

where $\{i,j,k\} = \{1,2,3\}$ and all even permutations, and

$$\varphi_\pm = \frac{1 \pm \sqrt{5}}{2}$$

Symmetry: H_3 Coxeter group (full symmetry group of the icosahedron). It has order 120.
The Hamiltonian is symmetric with respect to the transformation

$$x_i \longleftrightarrow x_j$$
$$\varphi_+ \longleftrightarrow \varphi_-$$

The ground state:

$$\Psi_0 = \Delta_1^{\nu}\Delta_2^{\nu}\exp\left(-\frac{\omega}{2}\sum_{k=1}^{3}x_k^2\right) \ , \quad E_0 = \frac{3}{2}\omega(1+10\nu)$$

where

$$\Delta_1 = \prod_{k=1}^{3} x_k$$

$$\Delta_2 = \prod_{\{i,j,k\}}\prod_{\mu_{1,2}=0,1}\left[x_i + (-1)^{\mu_1}\varphi_+ x_j + (-1)^{\mu_2}\varphi_- x_k\right]$$

$$h_{H_3} = -2(\Psi_0)^{-1}(\mathcal{H}_{H_3} - E_0)(\Psi_0)$$

New spectral problem arises

$$h_{H_3}\phi(x) = -2\epsilon\phi(x)$$

New variables ($x_{1,2,3} \to \tau_{1,2,3}$):

$$
\begin{aligned}
\tau_1 &= x_1^2 + x_2^2 + x_3^2 \\
\tau_2 &= -\frac{3}{10}\left(x_1^6 + x_2^6 + x_3^6\right) + \frac{3}{10}(2 - 5\varphi_+)\left(x_1^2 x_2^4 + x_2^2 x_3^4 + x_3^2 x_1^4\right) \\
&\quad + \frac{3}{10}(2 - 5\varphi_-)\left(x_1^2 x_3^4 + x_2^2 x_1^4 + x_3^2 x_2^4\right) - \frac{39}{5} \\
\tau_3 &= \frac{2}{125}\left(x_1^{10} + x_2^{10} + x_3^{10}\right) + \frac{2}{25}(1 + 5\varphi_-)\left(x_1^8 x_2^2 + x_2^8 x_3^2 + x_3^8 x_1^2\right) \\
&\quad + \frac{2}{25}(1 + 5\varphi_+)\left(x_1^8 x_3^2 + x_2^8 x_1^2 + x_3^8 x_2^2\right) \\
&\quad + \frac{4}{25}(1 - 5\varphi_-)\left(x_1^6 x_2^4 + x_2^6 x_3^4 + x_3^6 x_1^4\right) \\
&\quad + \frac{4}{25}(1 - 5\varphi_+)\left(x_1^6 x_3^4 + x_2^6 x_1^4 + x_3^6 x_2^4\right) \\
&\quad - \frac{112}{25}\left(x_1^6 x_2^2 x_3^2 + x_2^6 x_3^2 x_1^2 + x_3^6 x_1^2 x_2^2\right) \\
&\quad + \frac{212}{25}\left(x_1^2 x_2^4 x_3^4 + x_2^2 x_3^4 x_1^4 + x_3^2 x_1^4 x_2^4\right)
\end{aligned}
$$

$$h_{H_3} = \sum_{i,j=1}^{3} A_{ij} \frac{\partial^2}{\partial \tau_i \partial \tau_j} + \sum_{j=1}^{3} B_j \frac{\partial}{\partial \tau_j}$$

$$\begin{aligned}
A_{11} &= 4\tau_1 \\
A_{12} &= 12\tau_2 \\
A_{13} &= 20\tau_3 \\
A_{22} &= -\frac{48}{5}\tau_1^2\tau_2 + \frac{45}{2}\tau_3 \\
A_{23} &= \frac{16}{15}\tau_1\tau_2^2 - 24\tau_1^2\tau_3 \\
A_{33} &= -\frac{64}{3}\tau_1\tau_2\tau_3 + \frac{128}{45}\tau_2^3 \\
B_1 &= 6 + 60\nu - 4\omega\tau_1 \\
B_2 &= -\frac{48}{5}(1+5\nu)\tau_1^2 - 12\omega\tau_2 \\
B_3 &= -\frac{64}{15}(2+5\nu)\tau_1\tau_2 - 20\omega\tau_3
\end{aligned}$$

The Hamiltonian h_{H_3} preserves spaces

$$\mathcal{P}_n^{(1,2,3)} = \langle \tau_1^{n_1}\tau_2^{n_2}\tau_3^{n_3} | 0 \le n_1 + 2n_2 + 3n_3 \le n \rangle \ , \quad n \in \mathbf{N}$$

$\Rightarrow$ characteristic vector is (1,2,3), they form an *infinite flag*
The spectra:

$$\epsilon_{p_1,p_2,p_3} = 2\omega(p_1 + 3p_2 + 5p_3) \ , \quad p_i = 0, 1, 2, \dots$$

Degeneracy: $\qquad p_1 + 3p_2 + 5p_3 = \text{integer}$

Anisotropic harmonic oscillator with ratios 1:3:5

Eigenfunctions $\phi_{n,i}$ of h_{H_3} are elements of $\mathcal{P}_n^{(1,2,3)}$

$\mathcal{P}^{(3)}_{n,(1,2,3)}$ is finite-dimensional representation space of a Lie algebra of differential operators

We call this algebra $h^{(3)}$. It is infinite-dimensional but finitely generated

(with 30 generating elements of 1^{st}, 2^{nd} and 3^{rd} orders, they form 10 Abelian subalgebras and one Cartan type)

The H_4 integrable model

$$\mathcal{H}_{H_4} = \frac{1}{2}\sum_{k=1}^{4}\left[-\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 + \frac{\nu(\nu-1)}{x_k^2}\right] + 2\nu(\nu-1)\sum_{\mu_{2,3,4}=0,1}\frac{1}{[x_1 + (-1)^{\mu_2}x_2 + (-1)^{\mu_3}x_3 + (-1)^{\mu_4}x_4]^2}$$

$$+2\nu(\nu-1)\sum_{\{i,j,k,l\}}\sum_{\mu_{1,2}=0,1}\frac{1}{[x_i + (-1)^{\mu_1}\varphi_+ x_j + (-1)^{\mu_2}\varphi_- x_k + 0\cdot x_l]^2}$$

where $\{i,j,k,l\} = \{1,2,3,4\}$ and all even permutations.

$$\varphi_\pm = \frac{1\pm\sqrt{5}}{2}$$

Symmetry: H_4 Coxeter group (the symmetry group of the *600-cell*. It has order 14400.

The Hamiltonian is symmetric wrt $x_i \longleftrightarrow x_j$, $\varphi_+ \longleftrightarrow \varphi_-$

The ground state function and its eigenvalue

$$\Psi_0 = \Delta_1^{\nu} \Delta_2^{\nu} \Delta_3^{\nu} \exp\left(-\frac{\omega}{2} \sum_{k=1}^{4} x_k^2 \right) \ , \quad E_0 = 2\omega(1 + 30\nu)$$

where

$$\Delta_1 = \prod_{k=1}^{4} x_k$$

$$\Delta_2 = \prod_{\mu_{2,3,4}=0,1} [x_1 + (-1)^{\mu_2} x_2 + (-1)^{\mu_3} x_3 + (-1)^{\mu_4} x_4]$$

$$\Delta_3 = \prod_{\{i,j,k,l\}} \prod_{\mu_{1,2}=0,1} [x_i + (-1)^{\mu_1} \varphi_+ x_j + (-1)^{\mu_2} \varphi_- x_k + 0 \cdot x_l]$$

Make a gauge rotation of the Hamiltonian

$$h_{H_4} = -2(\Psi_0)^{-1}(\mathcal{H}_{H_4} - E_0)(\Psi_0) \ .$$

and introduce new variables $\tau_{1,2,3,4}$ as some polynomials in x of degrees 2,12,20,10 (degrees of H_4).

The Hamiltonian takes the algebraic form

$$h_{H_4} = \sum_{i,j=1}^{4} A_{ij} \frac{\partial^2}{\partial \tau_i \partial \tau_j} + \sum_{j=1}^{4} B_j \frac{\partial}{\partial \tau_j}$$

$$A_{11} = 4\ \tau_1\ ,\ A_{12} = 24\ \tau_2\ ,\ A_{13} = 40\ \tau_3\ ,\ A_{14} = 60\ \tau_4$$

$$A_{22} = 88\ \tau_1\tau_3 + 8\ \tau_1^5\tau_2\ ,\ A_{23} = -4\ \tau_1^3\tau_2^2 + 24\ \tau_1^5\tau_3 - 8\ \tau_4$$

$$A_{24} = 10\ \tau_1^2\tau_2^3 + 60\ \tau_1^4\tau_2\tau_3 + 40\ \tau_1^5\tau_4 - 600\ \tau_3^2$$

$$A_{33} = -\frac{38}{3}\ \tau_1\tau_2^3 + 28\ \tau_1^3\tau_2\tau_3 - \frac{8}{3}\ \tau_1^4\tau_4$$

$$A_{34} = 210\ \tau_1^2\tau_2^2\tau_3 + 60\ \tau_1^3\tau_2\tau_4 - 180\ \tau_1^4\tau_3^2 + 30\ \tau_2^4$$

$$A_{44} = -2175\ \tau_1\tau_2^3\tau_3 - 450\tau_1^2\tau_2^2\tau_4 - 1350\ \tau_1^3\tau_2\tau_3^2 - 600\ \tau_1^4\tau_3\tau_4$$

$$\begin{aligned}
B_1 &= 8(1+30\nu) - 4\omega\tau_1 \\
B_2 &= 12(1+10\nu)\ \tau_1^5 - 24\omega\tau_2 \\
B_3 &= 20(1+6\nu)\ \tau_1^3\tau_2 - 40\omega\tau_3 \\
B_4 &= 15(1-30\nu)\ \tau_1^2\tau_2^2 - 450(1+2\nu)\ \tau_1^4\tau_3 - 60\omega\tau_4
\end{aligned}$$

The algebraic operator h_{H_4} preserves subspaces

$$\mathcal{P}_n^{(1,5,8,12)} = \langle \tau_1^{n_1}\tau_2^{n_2}\tau_3^{n_3}\tau_4^{n_4} | 0 \le n_1 + 5n_2 + 8n_3 + 12n_4 \le n \rangle\ , \quad n \in \mathbf{N}$$

$\Rightarrow$ characteristic vector is (1,5,8,12), they form flag

$$\epsilon_{k_1,k_2,k_3,k_4} = 2\omega(k_1 + 6k_2 + 10k_3 + 15k_4)\ , \quad k_i = 0, 1, 2, \ldots$$

Degeneracy: $\qquad k_1 + 6k_2 + 10k_3 + 15k_4 = \text{integer}$

Anisotropic harmonic oscillator with ratios 1:6:10:15

Eigenfunctions $\phi_{n,i}$ of h_{H_4} are elements of $\mathcal{P}_n^{(1,5,8,12)}$

- For rational Hamiltonians for all exceptional root spaces $F_4, E_{6,7,8}$ (also trigonometric) and non-crystallographic $I_2(k)$, the eigenfunctions are polynomials in their invariants (in symmetric variables).

- Their hidden algebras are **new** infinite-dimensional but finite-generated algebras of differential operators. All of them have finite-dimensional invariant subspaces in polynomials.

- Generating elements of any such hidden algebra can be grouped in even number of (conjugated) Abelian algebras L_i, $\mathfrak{L}_\mathfrak{i}$ and one Lie algebra B.

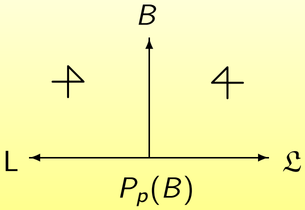


Figure: Triangular diagram relating the subalgebras L, $\mathfrak{L}$ and B. p is integer. It is a generalization of Gauss decomposition for semi-simple algebras ($p = 1$).

There are several solvable potentials in $1D$ generalized to D:

★
ES-case

$$\omega^2 r^2 + \frac{\gamma}{r^2} \qquad \to \qquad \omega^2 r^2 + \frac{\gamma(\Omega)}{r^2}$$

(generalization with discrete group of symmetry given by Weyl(Coxeter) group)

★
QES-case

$$\omega^2 r^2 + \frac{\gamma}{r^2} + ar^6 + br^4 \qquad \to \qquad \tilde{\omega}^2 r^2 + \frac{\tilde{\gamma}(\Omega)}{r^2} + ar^6 + br^4$$

(generalization with discrete group of symmetry given by Weyl(Coxeter) group)

ES-case

$$\frac{\gamma}{\sin^2(x)} \quad \to \quad \sum_{\alpha\in R_+} g_{|\alpha|} \frac{1}{\sin^2(\alpha\cdot x)}$$

where R_+ is a set of positive roots and $\mu_{|\alpha|}$ are coupling constants depending on the root length

(generalization with discrete group of symmetry given by Weyl(Coxeter) group + periodicity)

QES-cases

$$\frac{\gamma}{\sin^2(x)} + a\sin^4(x) + b\sin^2(x) \quad \to \quad ?$$

$$\frac{\gamma}{\sin^2(x)} + \frac{a}{\sin^6(x)} + \frac{b}{\sin^4(x)} \quad \to \quad ?$$

(generalization with discrete group of symmetry given by Weyl(Coxeter) group + periodicity)

Crucial moment of consideration:

Invariants of the discrete group of symmetry of the system taken as variables (space of orbits).

Happy Birthday, Francesco!