A Dirac-Moshinsky Oscillator coupled to an external field and its connection to quantum optics

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Francesco Calogero
on the Occasion of his 75th Birthday

Dirac-Moshinsky oscillator

The Dirac-Moshinsky oscillator was introduced in 1989 by Marcos Moshinsky and A. Szczepaniak as a solvable quantum relativistic model which in the non-relativistic limit corresponds to the hamiltonian of an harmonic oscillator with spin-orbit coupling term. It can be written as

$$i\hbar\frac{\partial|\Psi\rangle}{\partial t} = \left(c\boldsymbol{\alpha}\cdot(\mathbf{p}+im\omega\beta\mathbf{r}) + mc^2\beta\right)|\Psi\rangle$$

in what follows we shall use the following dirac matrices

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{pmatrix} \qquad \boldsymbol{\beta} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ a vector operator with the usual Pauli matrices. and the state vector can be expressed as

$$|\Psi\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}$$

Dirac-Moshinsky oscillator

We can write the Hamiltonian in matrix form

$$H = \begin{pmatrix} mc^2\mathbf{1} & c\,\boldsymbol{\sigma}\cdot(\mathbf{r}+i\mathbf{p}) \\ c\,\boldsymbol{\sigma}\cdot(\mathbf{r}-i\mathbf{p}) & -mc^2\mathbf{1} \end{pmatrix}$$

to show that it couples $|\psi_1\rangle$ and ψ_2. Squaring it one obtains

$$\frac{E^2-m^2c^4}{c^2}|\psi_2\rangle = \left(p^2+m^2\omega^2r^2-3\hbar\omega mc^2-2mc^2\omega\boldsymbol{\sigma}\cdot\mathbf{L}\right)|\psi_2\rangle$$

in the non relativistic limit $E = mc^2+\varepsilon$ and the term in the left becomes approximately $2mc^2\varepsilon$, so the non-relativistic energy ε is eigenvalue.

$1 + 1$ DO

Let us consider now a DO in one spatial dimension. Here we only need two anticommuting matrices and we choose to write

$$H^{(1)} = -c\sigma_y(p + im\omega\sigma_z x) + mc^2\sigma_z$$

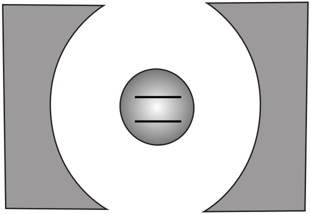
Using that the creation and annihilation operators $\sigma_\pm = (\sigma_x \pm \sigma_y)/2$ and $a = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{m\omega}$ we end up with

$$H^{(1)} = \sqrt{2mc^2\hbar\omega}\left(\sigma_+ a + \sigma_- a^\dagger\right) + mc^2\sigma_3$$

This is the well known Jaynes-Cummings Hamiltonian in Quantum optics.

Jaynes-Cummings model...

describes a two level atom interacting with one mode of the electromagnetic field in a cavity.



The connection with the DO:

- a and $a^\dagger$ with the creation and annihilation operators of one electromagnetic mode.
- $\sigma_\pm$ with the rise and lowering operators for a two level atom.
- $mc^2 \to \delta$: the detuning of the atomic transition from the mode frequency.
- $\sqrt{2mc^2\hbar\omega} \to \Omega$: the coupling between atom and mode.

Solution to the JC and $1+1$ DO

The solution to this system is well known

$$H_{\mathrm{JC}} = \Omega(\sigma_+ a + \sigma_+ a^\dagger) + \delta\sigma_z$$

One notices that there is a conserved quantity $I = a^\dagger a + \sigma_z/2$, so that the Hamiltonian can be diagonalized in the $|-,n\rangle$, $|+,n-1\rangle$ basis in blockdiagonal 2x2 matrices.

$$H_{\mathrm{JC}} = \left(\begin{array}{cc} \delta & \Omega\sqrt{n} \\ \Omega\sqrt{n} & -\delta \end{array}\right)$$

The eigenenergies are:

$$E_\pm = \pm\sqrt{\delta^2 + \Omega^2 n}$$

with the corresponding eigenstates (dressed states)

$$\begin{aligned} |\varphi_+\rangle &= \sin(\theta_n)|-,n\rangle + \cos(\theta_n)|+,n-1\rangle \\ |\varphi_-\rangle &= \cos(\theta_n)|-,n\rangle - \sin(\theta_n)|+,n-1\rangle \end{aligned}$$

with $\theta_n = \sqrt{\frac{E_+ - \delta}{E_+ + \delta}}$

2+1 DO

Now we need three anticommuting matrices and we choose

$$H^{(2)} = -c\sigma_x(p + imc^2\sigma_z x) - c\sigma_y(p + imc^2\sigma_z x) + mc^2\sigma_z$$

again, substituting with ladder operators of x and y and $\sigma_\pm$ we end up with

$$H^{(2)} = 2\sqrt{mc^2\hbar\omega}\left(\sigma_+ a_r + \sigma_- a_r^\dagger + mc^2\sigma_3\right)$$

where $a_r = (a_x + ia_y)/\sqrt{2}$, $a_r^\dagger = (a_x^\dagger - ia_y^\dagger)/\sqrt{2}$. These are creation and annihilation operators and fulfill the canonical commutation rule. $[a_r, a_r^\dagger] = 1$ The mapping onto JC works here too, one has to identify $a_r, a_r^\dagger \to a, a^\dagger$ of the cavity mode and $= 2\sqrt{mc^2\hbar\omega} \to \Omega$.

A. Bermudez *et al* Phys. Rev. A **76** 041801(2007)

3+1 DO

Going back to the $3+1$ case, we note that we can rewrite

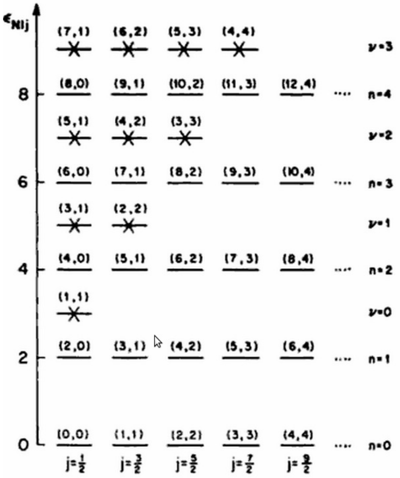
$$H = mc^2\Sigma_3 + \sqrt{2mc^2\hbar\omega}\left(\Sigma_-\boldsymbol{\sigma}\cdot\mathbf{a}^\dagger + \Sigma_+\boldsymbol{\sigma}\cdot\mathbf{a}\right) \tag{1}$$

$\sigma_\pm$ are raising and lowering operators, but we use capital letters here to distinguish them from the spin. $\mathbf{J}^2$ is a conserved quantity and additionally we have $\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\Sigma_3$ the eigenstates can be expressed as a combination of $|\pm\rangle$ and $|N(l,1/2)jm\rangle$ being the eigenstates of $\mathbf{J}^2$ and the $3D$ harmonic oscillator.

$$j = l \pm 1/2 \qquad N = 2n_r + l = n_x + n_y + n_z$$

Now we only need to find a good way of labeling these states. . .

Eigenstates



Taken from C. Quesne and M. Moshinsky, J. Phys. A **23** 2263 (1990)

The eigenstates in the $3+1$ case can be labeled as

$$\frac{E^2}{mc^2} = 1 + 2\hbar\omega \begin{cases} 2\nu + 3 & j = l - 1/2 \\ 2n & j = l + 1/2 \end{cases}$$

$$2\nu = N + j - 3/2 \qquad 2n = N - j - 1/2$$

Blocks of H in $3+1$

The Hamiltonian can be diagonalized by blocks with two separate cases

j=l+1/2

$$H(n) = \begin{pmatrix} mc^2 & \eta\sqrt{2n} \\ \eta\sqrt{2n} & -mc^2 \end{pmatrix}$$

$n = 0, 1, 2, \ldots$ for all j Connection with JC: $\eta\sqrt{2} \to \Omega$

$\eta = \sqrt{2mc^2\hbar\omega}$

J=l-1/2

$$H(\nu) = \begin{pmatrix} mc^2 & \eta\sqrt{2\nu+3} \\ \eta\sqrt{2\nu+3} & -mc^2 \end{pmatrix}$$

$\nu = j - \frac{1}{2}, j + \frac{1}{2}, j + \frac{3}{2} \ldots$ Connection with JC: only if one uses one pair of eigenstates.

Simplest interaction

The interaction is modeled as a potential that is summed to the total Hamiltonian

$$H \rightarrow H + \Phi$$

The simplest interaction we could possible imagine is a two level system which we call it simple a field. We want to conserve integrability, so we chose to be of the form
Where A represents for each of the dimensional cases

$$\Phi = \chi(T_- A^\dagger + T_+ A) + \gamma T_3$$

where

- 1+1: $A = a$
- 2+1: $A = a_r$
- 3+1: $A = \boldsymbol{\sigma} \cdot \mathbf{a}$

Now the constant of motion for each case is given by

$$I = A^\dagger A + 1/2(\sigma_3 + T_3)$$

The coupled Hamiltonian

The full Hamiltonian with interaction is given by

$$H = \eta(\Sigma_- A^\dagger + \Sigma_+ A) + \chi(T_- A^\dagger + \sigma_+ A) + mc^2\Sigma_3 + \gamma T_3$$

using the basis where $I = A^\dagger A + 1/2(\Sigma_3 + T_3)$

$$|n+1\rangle|{-}{-}\rangle \quad |n\rangle|{+}{-}\rangle \quad |n\rangle|{-}{+}\rangle \quad |n-1\rangle|{+}{+}\rangle$$

The Hamiltonian is now block diagonal with its blocks given by 4x4 matrices

$$H^{(n)} = \begin{pmatrix} -mc^2-\gamma & \chi & \eta\sqrt{n+1} & 0 \\ \xi\sqrt{n+1} & \gamma - mc^2 & 0 & \eta\sqrt{n} \\ \eta\sqrt{n+1} & 0 & mc^2-\gamma & \chi\sqrt{n} \\ 0 & \eta\sqrt{n} & \chi\sqrt{n} & mc^2+\gamma \end{pmatrix}.$$

(Actually this Hamiltonian can represent two two-level atoms in a cavity)

Entanglemennt with the field

We use a product initial state formed by an eigenstate of the DO and a state of the field

$$|\Psi(t=0)\rangle = |D0\rangle(\cos(\alpha)|+\rangle + \sin(\alpha)|-\rangle)$$

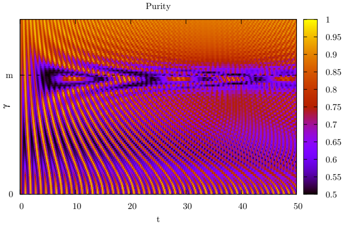
Consider the reduced density matrix for the external field, which can be obtained by tracing over the DO degrees of freedom

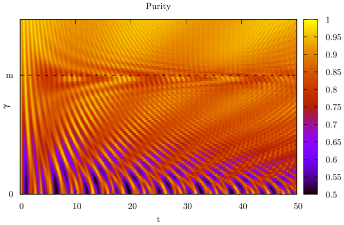
$$\rho = \mathrm{Tr}_{\mathrm{DO}}\{\varrho(t)\}$$

where $\varrho = |\Psi(t)\rangle\langle\Psi(t)|$. We evaluate the entanglement of the DO with the field using the purity, which can be obtained as

$$P = \mathrm{Tr}\{\rho^2\}$$

Some results





The parameters are $m = 3.2$, $c = \hbar = 1$ $\chi = 1.2$ To the left $\alpha = \pi/4$. To the right $\alpha = 0$. $n = 0$

Mapping the DMO onto Jaynes-Cummings model

DO coupled to an external field

Two atoms in a cavity

Conclusions

Two atoms inside a cavity

The last Hamiltonian is equivalent to the one of two atoms inside a cavity. This is a simple model that we shall use to study the evolution of entanglement in two different aspects:

- As a possible resource for implementing quantum information protocols. For this, entanglement is our allied.
- Entanglemnt of a central system to an environment, *i.e.* decoherence. This is an obstacle in quantum information protocols.

Here we distinguish between the two atoms (central system) and the cavity (environment).

Two interacting atoms in a cavity

We consider the Hamiltonian of two interacting atoms coupled to a cavity mode

$$H = \quad \sum_{j=1}^{2} \left\{ \delta_j \sigma_z^{(j)} + g_j \left(a\, \sigma_+^{(j)} + a^\dagger \sigma_-^{(j)} \right) \right\} + 2\kappa \left(\sigma_-^{(1)} \sigma_+^{(2)} + \sigma_+^{(1)} \sigma_-^{(2)} \right) + J \sigma_z^{(1)} \sigma_z^{(2)}$$

We consider the possibility of having different atoms, altough in this talk we will restric ourselves to identical atoms.
The number of excitations is conserved

$$I = N + \frac{1}{2} \left(\sigma_z^{(1)} + \sigma_z^{(2)} \right) \tag{2}$$

The basis where I is diagonal

$$|\phi_1^{(n)}\rangle = |n+1\rangle |--\rangle \qquad |\phi_2^{(n)}\rangle = |n\rangle |-+\rangle$$
$$|\phi_3^{(n)}\rangle = |n\rangle |+-\rangle \qquad |\phi_4^{(n)}\rangle = |n-1\rangle |++\rangle.$$

Diagonalization of the Hamiltonian

In the previous basis where the number of excitations is conserved, the Hamiltonian is block diagonal

$$H = \begin{pmatrix} H^{(0)} & 0 & 0 & \dots \\ 0 & H^{(1)} & 0 & \dots \\ 0 & 0 & H^{(2)} & \\ \vdots & \vdots & & \ddots \end{pmatrix},$$

with each block given by

$$H^{(n)} = \begin{pmatrix} J-\delta_1-\delta_2 & g_2\sqrt{n+1} & g_1\sqrt{n+1} & 0 \\ g_2\sqrt{n+1} & \delta_2-\delta_1-J & 2\kappa & g_1\sqrt{n} \\ g_1\sqrt{n+1} & 2\kappa & \delta_1-\delta_2-J & g_2\sqrt{n} \\ 0 & g_1\sqrt{n} & g_2\sqrt{n} & J+\delta_1+\delta_2 \end{pmatrix}.$$

Time evolution

We begin with the product state

$$|\Psi_0\rangle = |n\rangle \left(\cos(\alpha)|-+\rangle + \sin(\alpha)|+-\rangle\right).$$

at a time t the state vector can be found in

$$|\Psi(t)\rangle = \sum_{l=1}^{4} B_l^{(n)}(t)|\phi_l^{(n)}\rangle$$

we trace over the oscillator degrees of freedom to get the reduced density matrix for the two atoms: $\rho = \mathrm{Tr}_n \{|\Psi(t)\rangle\langle\Psi(t)|\} =$

$$= \begin{pmatrix} |B_1^{(n)}|^2 & 0 & 0 & 0 \\ 0 & |B_2^{(n)}|^2 & \left(B_3^{(n)}\right)^* B_2^{(n)} & 0 \\ 0 & \left(B_2^{(n)}\right)^* B_3^{(n)} & |B_3^{(n)}|^2 & 0 \\ 0 & 0 & 0 & |B_4^{(n)}|^2 \end{pmatrix}.$$

Entanglement measures

To measure the entanglement between the two atoms (central system) and the cavity (environment) we use the purity $P = \mathrm{Tr}\{\rho\}$ and found

$$P = |B_1^{(n)}|^4 + |B_4^{(n)}|^4 + \left(1 - |B_1^{(n)}|^2 - |B_4^{(n)}|^2\right)^2 .$$

To measure the entanglement between the atoms we use the Concurrence $C(\rho) = \mathrm{Max}\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where λ_j are the eigenvalues of $\left(\rho\, \sigma_y^{(1)} \sigma_y^{(2)}\, \rho^*\, \sigma_y^{(1)} \sigma_y^{(2)}\right)^{1/2}$ in non-increasing order. In this case we find

$$C(\rho) = \mathrm{Max}\left\{0,\; 2|B_2^{(n)}||B_3^{(n)}| - 2|B_1^{(n)}||B_4^{(n)}|\right\}.$$

Concurrence and purity in time

$$
\begin{aligned}
C(t) &= \mathrm{Max}\left\{0, \sqrt{(\sin(2\alpha) - F(t))^2 + \cos^2(2\alpha) G^2(t)} - \beta_n F(t)\right\} \\
P(t) &= 1 - 2F(t) + \gamma_n F^2(t).
\end{aligned}
$$

where

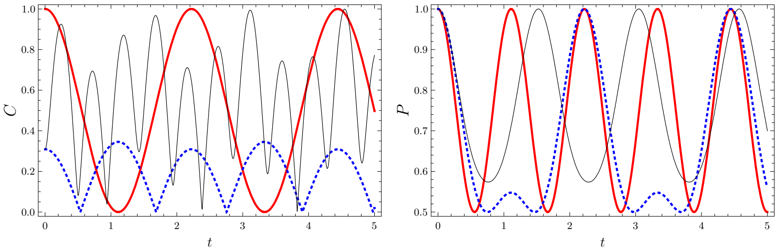
$$
\begin{aligned}
F(t) &= \tfrac{2n+1}{\omega_n^2}(1 + \sin(2\alpha)) \sin^2(\omega_n t) \\
G(t) &= \tfrac{(\kappa - J)\cos((J+3\kappa)t)\sin(\omega_n t)}{\omega_n} + \\
&\quad \sin((J+3\kappa)t)\cos(\omega_n t),
\end{aligned}
$$

and

$$
\omega_n = \sqrt{4n + 2 + (\kappa - J)^2}
$$

$$
\beta_n = \sqrt{\tfrac{4n^2+4n}{4n^2+4n+1}}, \quad \gamma_n = \tfrac{6n^2+6n+2}{4n^2+4n+1},
$$

Some results in time domain for $n = 0$ and identical atoms



In red, non-interacting atoms with an initial state determined by $\alpha = \pi/4$ *i.e.* a maximally entangled pure state. In blue the behavior for non-interacting atoms with an initial pure, but not maximally entangled state with $\alpha = \pi/20$. In black, the curve for two interacting atoms with the same initial state as in the blue dashed curve and $\kappa = 1.5$ and $J = 0$.

CP-Plane

We will now proceed to visualize the dynamics in the plane concurrence vs purity. Analytic expressions can be found in the case without interaction

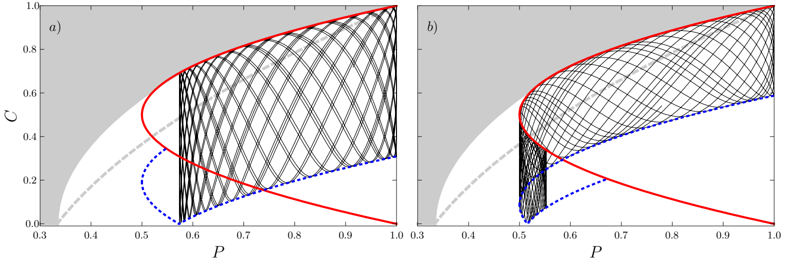
$$C_{\pm}^{(n)}(P;\alpha) \quad = \mathrm{Max}\left\{0, \left|\sin(2\alpha) - f_{\pm}^{(n)}(P)\right| - \beta_n \, f_{\pm}^{(n)}(P)\right\}$$

with

$$f_{\pm}^{(n)}(P) = \frac{1 \pm \sqrt{1+\gamma_n(P-1)}}{\gamma_n}.$$

we'll show that these curves form a boundary for the interacting case.

CP-Plane $n = 0$ and identical atoms

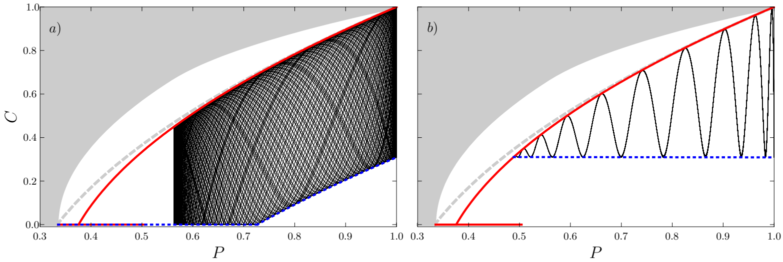


a b

In red non-interacting atoms with $\alpha = \pi/4$ *i.e.* a maximally entangled pure state. In blue, non-interacting atoms with an initial pure, but not maximally entangled state *a*) $\alpha = \pi/20$ and *b*) $\alpha = \pi/10$. In black, interacting atoms with $\alpha = \pi/20$ parametrized by time up to $t = 20$. a) $\kappa = 1.5$ and $J = 0$. b) $\kappa = 1.5$ and $J = 0.87$. The gray area indicates *CP* combinations that can not be obtained in physical states and its lower frontier corresponds to the *maximally entangled mixed states*.

$$C_{\pm}^{(0)}(P; \pi/4) = \frac{1}{2}\left(1 \mp \sqrt{2P - 1}\right)$$

CP-Plane $n = 5$



a b

$n = 5$. a) $\alpha = \pi/20$ for the black and blue line and $\kappa = 5.7$ and $J = 0.2$ for the black line. b) $\alpha = -\pi/20$ for the black and blue line and $\kappa = J = 5\sqrt{4 \times 5 + 2}$ for the black line.

Summary and conclusions

- DMO
  - 1+1 and 2+1 DMO can be exactly mapped onto the JC model.
  - In the 3+1 case the degenerate part can be mapped.
  - If one uses only one pair of DMO eigenstates $n+1$ the resulting Hamiltonian is a 2x2 matrix that can be connected to a corresponding JC model.
- DMO coupled to an isospin field
  - Choosing the interaction carefully, the system retains solvability.
  - The resulting model can be connected to a double JC model, or two two-level atoms inside a cavity.
- Two atoms in a cavity
  - The model is solvable and allows closed results for purity and concurrence.
  - we can characterize the dynamics in the CP-Plane.
  - is simple model to study entanglement, both as a resource and as a source of decoherence.

The results are taken from the following papers:

- JM Torres E. Sadurni and TH Seligman 2010 J. Phys. A: Math. Theor. 43 192002
- E Sadurní JM Torres and TH Seligman 2010 J. Phys. A: Math. Theor. 43 285204
- JM Torres, E Sadurni and TH Seligman, arXiv:1010.5229,
  in: Proceedings of Symmetries in Nature Symposium in Memoriam Marcos Moshinsky, Cuernavaca 2010, AIP Proceedings (in press).

Thank you for your attention.