Polynomial integrable differential equations and non-associative algebras.

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Symmetry approach to classification of integrable PDEs.

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Was developed by: A.Shabat, A.Zhiber, N.Ibragimov, A.Fokas, V.Sokolov, S.Svinolupov, A.Mikhailov, R.Yamilov, V.Adler, P.Olver, J.Sanders, J.P.Wang, V.Novikov, A.Meshkov, D.Demskoy, H.Chen, Y.Lee, C.Liu, I.Khabibullin, B.Magadeev, R.Heredero, V.Marikhin ...

Definition. PDE is integrable if it possesses infinitely many generalized infinitesimal symmetries.

Symmetries for ODEs.

Suppose we have a dynamical system

$$\frac{d\,u_i}{dt} = F_i(u_1, \ldots, u_n), \qquad i = 1, \ldots, n\,. \tag{1}$$

Definition. The dynamical system

$$\frac{d\,u_i}{d\tau} = G_i(u_1, \ldots, u_n), \qquad i = 1, \ldots, n\,. \tag{2}$$

is called (infinitesimal) symmetry for (1) iff (1) and (2) are compatible.

It means that for any initial data $\mathbf{u}_0$ there exists a common solution $\mathbf{u}(t, \tau)$ of systems (1) and (2) such that $\mathbf{u}(0, 0) = \mathbf{u}_0$.

Evolution (1+1)-dimensional equations.

Consider evolution equation

$$u_t = F(u, u_x, u_{xx}, \ldots, u_n), \qquad u_i = \frac{\partial^i u}{\partial x^i}. \qquad (3)$$

The generalized (higher) symmetry is an evolution equation

$$u_\tau = G(u, u_x, u_{xx}, \ldots, u_m),$$

that is compatible with (3).

Example 1. For any m equation $u_\tau = u_m$ is a symmetry for $u_t = u_n$.

Example 2. The Burgers equation

$$u_t = u_{xx} + 2uu_x$$

has the following third order symmetry

$$u_\tau = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x.$$

Example 3. The simplest higher symmetry for the Korteweg-de Vries equation

$$u_t = u_{xxx} + 6uu_x$$

has the following form

$$u_\tau = u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x.$$

Why integrable equations possess higher symmetries?

"Explanation". A linear equation has infinitely many higher symmetries. Integrable nonlinear equation is related to a linear one by some transformation. The same transformation produces higher symmetries for nonlinear equation starting from symmetries of the linear one.

For instance, the Burgers equation is integrable because of the Cole-Hopf substitution

$$u = \frac{v_x}{v},$$

which relates the equation to $v_t = v_{xx}$. Moreover, the same substitution maps the third order symmetry of the Burgers equation to

$$v_\tau = v_{xxx},$$

etc.

Some necessary integrability conditions, which do not depend on symmetry orders were found by Ibragimov-Shabat-VS. It was proved by Svinolupov-VS that the same conditions fulfiled if equation possesses infinitely many local conservation laws.

The first classification result in frames of the symmetry approach was:

Theorem. (Shabat-Zhiber 1979)
Nonlinear hyperbolic equation of the form

$$u_{xy} = F(u)$$

possesses higher symmetries iff (up to scalings and shifts)

$F(u) = e^u$, $F(u) = e^u + e^{-u}$, or $F(u) = e^u + e^{-2u}$.

The complete classification of integrable hyperbolic equations of the form

$$u_{xy} = F(u, u_x, u_y)$$

is an open problem till now.

Example:

$$u_{xy} = S(u)\sqrt{1 - u_x^2}\sqrt{1 - u_y^2},$$

$$S'' - 2S^3 + c\,S = 0;$$

Theorem (Svinolupov-VS 1982). A complete list (up to "almost invertible" transformations) of equations of the form

$$u_t = u_{xxx} + f(u_{xx}, u_x, u) \qquad (4)$$

with infinite hierarchy of conservation laws can be written as:

$$\begin{aligned}
u_t &= u_{xxx} + u\,u_x, && \text{KdV}\\
u_t &= u_{xxx} + u^2\,u_x, && \text{mKdV}\\
u_t &= u_{xxx} - \frac{1}{2}u_x^3 + (\alpha e^{2u} + \beta e^{-2u})u_x, && \text{CD1}\\
u_t &= u_{xxx} - \frac{1}{2}Q''\,u_x + \frac{3}{8}\frac{(Q-u_x^2)_x^2}{u_x\,(Q-u_x^2)}, && \text{CD2}\\
u_t &= u_{xxx} - \frac{3}{2}\,\frac{u_{xx}^2 + Q(u)}{u_x} && \text{KN},
\end{aligned}$$

where $Q'''''(u) = 0$.

All integrable equations of the form

$$u_t = F(u_{xx}, u_x, u, x, t)$$

were listed by Svinolupov 1985 and by VS-Svinolupov 1991.

The answer is:

$$u_t = u_2 + 2uu_x + h(x),$$
$$u_t = u^2 u_2 - \lambda x u_1 + \lambda u$$
$$u_t = u^2 u_2 + \lambda u^2$$
$$u_t = u^2 u_2 - \lambda x^2 u_1 + 3\lambda x u$$

This is a complete list up to the contact transformations

$$\hat{x} = \varphi(x, u, u_1), \qquad \hat{u} = \psi(x, u, u_1),$$
$$\hat{u}_i = \left(\tfrac{1}{D_x(\varphi)} D_x\right)^i (\psi),$$

where

$$D_x(\varphi) \frac{\partial \psi}{\partial u_1} = D_x(\psi) \frac{\partial \varphi}{\partial u_1}.$$

All equations of the form

$$u_t = u_5 + F(u_4, u_3, u_2, u_1, u),$$

possessing higher conservation laws were found by Drinfeld-VS-Svinolupov 1985.

Examples: Well-known equations

$$u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1,$$
$$u_t = u_5 + 5uu_3 + \tfrac{25}{2}u_1u_2 + 5u^2u_1$$
$$u_t = u_5 + 5(u_1 - u^2)u_3 + 5u_2^2 - 20uu_1u_2$$
$$-5u_1^3 + 5u^4u_1$$

A new equation

$$\begin{aligned} u_t \;\; = \;\; & u_5 + 5(u_2 - u_1^2 + \lambda_1 e^{2u} - \lambda_2^2 e^{-4u})\, u_3 \\ & -5u_1u_2^2 + 15(\lambda_1 e^{2u} + 4\lambda_2^2 e^{-4u})\, u_1u_2 + u_1^5 \\ & -90\lambda_2^2 e^{-4u}\, u_1^3 + 5(\lambda_1 e^{2u} - \lambda_2^2 e^{-4u})^2\, u_1 \end{aligned}$$

Classification of systems.

The most significant work has been done by Mikhailov-Shabat-Yamilov 1987. All systems of the form

$$u_t = u_2 + F(u, v, u_1, v_1),$$

$$v_t = -v_2 + G(u, v, u_1, v_1)$$

possessing higher conservation laws, were listed.

Example 1: Well-known NLS-equation

$$u_t = u_2 + u^2 v,$$

$$v_t = -v_2 - v^2 u,$$

Example 2. The Landau-Lifshitz equation (after stereographic projection)

$$u_t = u_2 - \frac{2u_1^2}{u+v} - \frac{4(\,p(u,v)u_1 + r(u)v_1\,)}{(u+v)^2}$$

$$v_t = -v_2 + \frac{2v_1^2}{u+v} - \frac{4(\,p(u,v)v_1 + r(-v)u_1\,)}{(u+v)^2},$$

where

$$r(y) = c_4y^4 + c_3y^3 + c_2y^2 + c_1y + c_0$$

and

$$p(u,v) = 2c_4u^2v^2 + c_3(uv^2 - vu^2) - 2c_2uv + c_1(u-v) + 2c_0.$$

Algebraic structures and polynomial integrable models.
(Svinolupov)

The Burgers equation is given by

$$u_t = u_{xx} + 2uu_x.$$

Consider the following multi-component generalization:

$$u^i_t = u^i_{xx} + 2C^i_{jk}u^k u^j_x + A^i_{jkm}u^k u^j u^m,$$

where $\quad i, j, k = 1, \ldots, N$

Theorem 1 . This system has generalized symmetries iff

$$A^i_{jkm} = \tfrac{1}{3}(C^i_{jr}C^r_{km} + C^i_{kr}C^r_{mj} + C^i_{mr}C^r_{jk} \\ -C^i_{rj}C^r_{km} - C^i_{rk}C^r_{mj} - C^i_{rm}C^r_{jk}),$$

and

$$C^i_{jr}C^r_{km} - C^i_{kr}C^r_{jm} = C^r_{jk}C^i_{rm} - C^r_{kj}C^i_{rm}$$

for any i, j, k, m (summation w.r.t. r).

The latter formula means that C^i_{jk} are structural constants of a left-symmetric algebra!

The algebraic form of the system is

$$U_t = U_{xx} + 2U \circ U_x + U \circ (U \circ U) - (U \circ U) \circ U,$$

where $\circ$ denotes the multiplication in a left-symmetric algebra A, $\quad U = \sum_k u^k e_k$ and

$$e_j \circ e_k = C^i_{jk} e_i.$$

Definition of left-symmetric algebra:

$$As(X, Y, Z) = As(Y, X, Z),$$

where

$$As(X, Y, Z) = (X \circ Y) \circ Z - X \circ (Y \circ Z).$$

Example of left-symmetric algebra (VS). The set of all N-dimensional vectors w.r.t.

$$X \circ Y =< X, C > Y+ < X, Y > C,$$

where C is a fixed (constant) vector.

Examples of corresponding integrable systems: Svinolupov-VS 1994.

The matrix Burgers equation

$$u_t = u_{xx} + uu_x;$$

the vector Burgers equation

$$\begin{aligned} u_t = u_{xx} + 2 < u, u_x > C + 2 < C, u > u_x + \\ < u, u >< C, u > C- < u, u >< C, C > u; \end{aligned}$$

One more example of left-symmetric algebra (I.Golubchik-VS).

Let A be associative algebra and $R : A \to A$ satisfies the modified classical Yang-Baxter equation

$$R([R(x), y] - [R(y), x]) = [x, y] + [R(x), R(y)].$$

Then the multiplication

$$x \circ y = [R(x), y] - (xy + yx)$$

is left-symmetric.

The KdV equation is given by

$$u_t = u_{xxx} + u u_x.$$

Theorem 2. If C^i_{jk} are structural constants of any Jordan algebra then the KdV-type system

$$u^i_t = u^i_{xxx} + C^i_{jk} u^k u^j_x, \qquad i, j, k = 1, \ldots, N$$

possesses higher symmetries.

The algebraic form

$$U_t = U_{xxx} + U \circ U_x,$$

where $\circ$ denotes the multiplication in a Jordan algebra A.

Definition of Jordan algebra:

$$X \circ Y = Y \circ X, \qquad X^2 \circ (Y \circ X) = (X^2 \circ Y) \circ X.$$

If $*$ is a multiplication in an associative algebra then $X \circ Y = X * Y + Y * X$ is a Jordan operation.

Examples of simple Jordan algebras.
a) The set of all $N \times N$ matrices w.r.t.

$$X \circ Y = XY + YX$$

b) The set of all N-dimensional vectors w.r.t.

$$X \circ Y = < X, C > Y + < Y, C > X - < X, Y > C.$$

The corresponding integrable systems:

the matrix KdV-equation:

$$u_t = u_{xxx} + uu_x + u_x u;$$

the vector KdV equation (Svinolupov-VS):

$$u_t = u_{xxx} + < C, u > u_x + < C, u_x > u - < u, u_x > C;$$

Theorem 3. If C^i_{jkm} are structural constants of any Jordan triple system then the mKdV-type system

$$u^i_t = u^i_{xxx} + C^i_{jkm} u^k u^j u^m_x, \quad i,j,k = 1,\dots,N$$

possesses higher symmetries.

Theorem 4. If C^i_{jkm} are structural constants of any Jordan triple system then the nonlinear Schroedinger-type system

$$u^i_t = u^i_{xx} + C^i_{jkm} u^j v^k u^m, \qquad i,j,k = 1,\dots,N$$
$$v^i_t = -v^i_{xx} - C^i_{jkm} v^j u^k v^m$$

possesses higher symmetries.

Theorem 5. If C^i_{jkm} are structural constants of any Jordan triple system then the nonlinear derivative Schroedinger-type system

$$u^i_t = u^i_{xx} + C^i_{jkm} (u^j v^k u^m)_x, \qquad i,j,k = 1,\dots,N$$
$$v^i_t = -v^i_{xx} - C^i_{jkm} (v^j u^k v^m)_x$$

possesses higher symmetries.

Algebraic forms of the systems. The "Jordan" mKdV-equation

$$u_t = u_{xxx} + \{u, u, u_x\},$$

the "Jordan" nonlinear Schrödinger equation

$$u_t = u_{xx} + 2\{v, u, v\}, \quad v_t = -v_{xx} - 2\{u, v, u\},$$

The "Jordan" derivative nonlinear Schrödinger equation

$$u_t = u_{xx} + 2\{v, u, v\}_x, \quad v_t = -v_{xx} - 2\{u, v, u\}_x,$$

are integrable for any Jordan triple system.

Definition of Jordan triple system:

$$\{X,Y,Z\} = \{Z,Y,X\},$$

$$\{X,Y,\{V,W,Z\}\} - \{V,W,\{X,Y,Z\}\} = \\ \{\{X,Y,V\},W,Z\} - \{V,\{Y,X,W\},Z\}.$$

Examples of simple triple Jordan systems.

a) The set of all $N \times N$ matrices w.r.t.

$$\{X,Y,Z\} = XYZ + ZYX$$

b) The set of all N-dimensional vectors w.r.t.

$$\{X,Y,Z\} = < X,Y > Z + < Y,Z > X - < X,Z > Y.$$

c) The set of all N-dimensional vectors w.r.t.

$$\{X,Y,Z\} = < X,Y > Z + < Y,Z > X.$$

The corresponding integrable vector systems:

the matrix NLS equation

$$u_t = u_2 + 2\,uvu,$$
$$v_t = -v_2 - 2\,vuv;$$

the vector NLS equation 1 (Manakov)

$$u_t = u_2 + < u, v > u,$$
$$v_t = -v_2 - < u, v > v;$$

the vector NLS equation 2 (Kulish-Sklyanin)

$$u_t = u_2 + 2 < u, v > u - < u, u > v,$$
$$v_t = -v_2 - 2 < u, v > v + < v, v > u;$$

Inverse elements in Jordan triple systems and rational integrable models.

(Svinolupov - VS)

Let $\{X, Y, Z\}$ be a Jordan triple system,

$$u = \sum_k u^k e_k.$$

Let $\phi(u) = \sum_k \phi^k(u) e_k$ be a solution of the following overdetermined consistent system

$$\frac{\partial \phi}{\partial u^k} = -\{\phi,\ e_k,\ \phi\}. \tag{5}$$

Denote

$$\alpha_u(X, Y) = \{X, \phi(u), Y\}$$

$$\sigma_u(X, Y, Z) = \{X, \{\phi(u), Y, \phi(u)\}, Z\}.$$

If $\{X, Y, Z\} = \frac{1}{2}(XYZ + ZYX)$, then

$$\phi(u) = u^{-1}.$$

For

$\{X, Y, Z) =< X, Y > Z+ < Y, Z > X- < X, Z > Y$

we have

$$\phi(u) = \frac{u}{\|u\|^2}.$$

Class 3.1. Consider the equation

$$u_{xy} = \alpha_u(u_x, u_y) \tag{6}$$

In the matrix case this coincides with the equation of the principal chiral field

$$u_{xy} = \frac{1}{2}(u_x u^{-1} u_y + u_y u^{-1} u_x).$$

For this reason we will call (6) the Jordan chiral field equation.

It is easy to verify that (6) admits the following zero-curvature representation

$$\Psi_x = \frac{2}{(1-\lambda)} L_{u_x} \Psi, \qquad \Psi_y = \frac{2}{(1+\lambda)} L_{u_y} \Psi.$$

Here we denote by L_X the left multiplication operator: $L_X(Y) = \alpha_u(X, Y)$.

Class 3.2. Consider the following equation

$$u_t = u_{xxx} - 3\alpha_u(u_x, u_{xx}) + \frac{3}{2}\sigma_u(u_x, u_x, u_x)$$

The corresponding matrix and equation has the following form:

$$u_t = u_{xxx} - \frac{3}{2}u_x u^{-1} u_{xx} - \frac{3}{2}u_{xx} u^{-1} u_x + \frac{3}{2}u_x u^{-1} u_x u^{-1} u_x,$$

where $u(x,t)$ is an $m \times m$ matrix.

Class 3.2. The following integrable equations of the Schwartz-KdV type are given by

$$u_t = u_{xxx} - \frac{3}{2}\alpha_{u_x}(u_{xx}, u_{xx})$$

The matrix equation:

$$u_t = u_{xxx} - \frac{3}{2}u_{xx} u_x^{-1} u_{xx}$$

Class 3.3. The scalar representative of this class is the Heisenberg model

$$u_t = u_{xx} - \frac{2}{u+v}u_x^2, \qquad v_t = -v_{xx} + \frac{2}{u+v}v_x^2.$$

The corresponding Jordan coupled equations are given by

$$u_t = u_{xx} - 2\alpha_{u+v}(u_x, u_x), \;\; v_t = -v_{xx} + 2\alpha_{u+v}(v_x, v_x)$$

The matrix equation is of the form

$$u_t = u_{xx} - 2u_x(u+v)^{-1}u_x,$$

$$v_t = -v_{xx} + 2v_x(u+v)^{-1}v_x.$$

Equations of geometric type.

Consider multi-component systems of the form

$$u^i_t = u^i_{xxx} + a^i_{jk}(\vec{u}) u^j_x u^k_{xx} + b^i_{jks}(\vec{u}) u^j_x u^k_x u^s_x.$$

This class is invariant under point transformations: $\vec{v} = \vec{\Psi}(\vec{u})$. Under these transformations, the set of functions $a^i_{jk}(\vec{u})$ are transformed as components of an affine connection Γ.

It is convenient to rewrite the system as

$$u^i_t = u^i_{xxx} + 3\alpha^i_{jk} u^j_x u^k_{xx} +$$

$$\left(\frac{\partial \alpha^i_{km}}{\partial u^j} + 2\alpha^i_{jr}\alpha^r_{km} - \alpha^i_{rj}\alpha^r_{km} + \beta^i_{jkm} \right) u^j_x u^k_x u^m_x,$$

where $\beta^i_{jkm} = \beta^i_{kjm} = \beta^i_{mkj}$, i.e.

$$\beta(X, Y, Z) = \beta(Y, X, Z) = \beta(X, Z, Y)$$

for any vectors X, Y, Z. The set of functions β^i_{jkm} are transformed just as components of a tensor.

Let R and T be the curvature and torsion tensors of Γ.

In order to formulate classification results, we introduce the following tensor:

$$\sigma(X,Y,Z) = \beta(X,Y,Z) - \frac{1}{3}\delta(X,Y,Z) + \frac{1}{3}\delta(Z,X,Y),$$

where

$$\delta(X,Y,Z) = T(X,T(Y,Z)) + R(X,Y,Z) -$$

$$\nabla_X(T(Y,Z)).$$

It follows from the Bianchi identity that

$$\sigma(X,Y,Z) = \sigma(Z,Y,X).$$

Theorem. The system is integrable iff

$$\nabla_X[R(Y,Z,V)] = R(Y,X,T(Z,V)),$$

$$\nabla_X\left[\nabla_Y(T(Z,V)) - T(Y,T(Z,V)) - R(Y,Z,V)\right] = 0,$$

$$\nabla_X(\sigma(Y,Z,V)) = 0,$$

$$T(X,\sigma(Y,Z,V)) + T(Z,\sigma(Y,X,V)) +$$

$$+T(Y,\sigma(X,V,Z)) + T(V,\sigma(X,Y,Z)) = 0,$$

and

$$\sigma(X,\sigma(Y,Z,V),W) - \sigma(W,V,\sigma(X,Y,Z)) +$$

$$\sigma(Z,Y,\sigma(X,V,W)) - \sigma(X,V,\sigma(Z,Y,W)) = 0.$$

If $T = 0$, we have the symmetric space with covariantly constant deformation of a triple Jordan system.

In the case $T \neq 0$, a generalization of the symmetric spaces gives rise. We do not know wether such affine connected spaces have been considered by geometers.

Classification of integrable matrix evolution equations.

Olver and VS listed integrable non-abelian polynomial evolution equations having higher symmetries. One of the non-abelian lists:

$$u_t = u_{xxx} + 3u^2u_x + 3u_xu^2,$$
$$u_t = u_{xxx} + 3uu_{xx} - 3u_{xx}u - 6uu_xu,$$
$$u_t = u_{xxx} + 3u_x^2.$$

Second order systems of NLS- and DNLS-types also were listed and several new integrable models were found.

Examples.

$$u_t = u_{xx}+2(u+v)u_x, \qquad v_t = -v_{xx}+2v_x(u+v);$$

$$u_t = u_{xx} + 2u_xvu, \qquad v_t = -v_{xx} + 2vuv_x.$$

Matrix Painleve equations:

$$u_{xx} + 3u^2 = xE + C,$$

$$u_{xx} + 2u^3 + xu = \lambda E,$$

$$u_{xx} + \frac{1}{x}u_x = u_x u^{-1} u_x.$$

Classification of integrable matrix ODEs.

Polynomial non-abelian ODEs have been considered by Mikhailov-VS, 2000 and some partial classification results have been obtained.

For example, the following system

$$u_t = v^2, \qquad v_t = u^2$$

possesses infinitely many symmetries

$$u_{\tau_i} = P_i(u, v), \qquad v_{\tau_i} = Q_i(u, v)$$

and first integrals

$$\rho_i = \text{trace}\, R_i(u, v).$$

There exists two interesting integrable non-abelian equations containing arbitrary constant element C:

$$u_t = Cu^2 - u^2C$$

and

$$u_t = uCu^2 - u^2Cu.$$

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