On commuting pairs of Hamiltonians quadratic in momenta.

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Main concepts
of Hamiltonian mechanics.

Let $x_1, \ldots, x_m$ be the coordinates. Any Poisson bracket between functions $f(x_1, \ldots, x_m)$ and $g(x_1, \ldots, x_m)$ is given by

$$\{f,\, g\} = \sum_{i,j} P_{i,j}(x_1, \ldots, x_m) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j},$$

where $P_{i,j} = \{x_i, x_j\}$. The functions $P_{i,j}$ are not arbitrary since by definition

$$\{f, g\} = -\{g, f\},$$

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0.$$

The corresponding dynamical systems are

$$\frac{dx_i}{dt} = \{H, x_i\},$$

where H is a Hamiltonian function. A function K is an integral of motion for this system iff $\{K,\, H\} = 0$.

If $\{J, f\} = 0$ for any f, then J is called a *Casimir function* of the Poisson bracket. The Casimir functions exist if the bracket is degenerate (i.e. $Det\, P = 0$).

The coordinates for the standard symplectic manifold are q_i and p_i, $i = 1, \ldots N$. The Poisson bracket is given by

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \qquad \{p_i, q_j\} = \delta_{i,j}.$$

The corresponding dynamical system has the usual Hamiltonian form

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}.$$

A change of variables is said to be *canonical* if it preserves this form of the bracket.

For the spinning tops the Hamiltonian structure is defined by a linear Poisson bracket, i.e. $P_{ij} = C^k_{ij} x_k$.

For the models of rigid body dynamics the Poisson bracket is given by the following $e(3)$ Poisson barcket:

$$\{M_i, M_j\} = \varepsilon_{ijk}\, M_k,$$

$$\{M_i, \gamma_j\} = \varepsilon_{ijk}\, \gamma_k, \qquad \{\gamma_i, \gamma_j\} = 0.$$

Here M_i and γ_i are components of 3-dimensional vectors $\mathbf{M}$ and $\mathbf{\Gamma}$, ε_{ijk} is the totally skew-symmetric tensor. This bracket has the two Casimir functions

$$J_1 = (\mathbf{M}, \mathbf{\Gamma}), \qquad J_2 = |\mathbf{\Gamma}|^2,$$

where $(\cdot, \cdot)$ stands for the standard dot product in $\mathbb{R}^3$.

For the Liouville integrability of the equations of motion only one additional integral functionally independent of the Hamiltonian and the Casimir functions is necessary.

For the Kirchhoff equations describing the motion of a rigid body in an ideal fluid there are classical integrable cases found by Clebsch and Steklov-Lyapunov. For these cases the Hamiltonian is of the form

$$H = a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 + \\ 2b_1\, M_1\gamma_1 + 2b_2\, M_2\gamma_2 + 2b_3\, M_3\gamma_3 + \\ c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2.$$

For the **Clebsch** case the coefficients a_i are arbitrary and the remaining parameters satisfy the following conditions

$$b_1 = b_2 = b_3,$$

$$\frac{c_1 - c_2}{a_3} + \frac{c_3 - c_1}{a_2} + \frac{c_2 - c_3}{a_1} = 0.$$

In the **Steklov-Lyapunov** case a_i are arbitrary and

$$\frac{b_1 - b_2}{a_3} + \frac{b_3 - b_1}{a_2} + \frac{b_2 - b_3}{a_1} = 0,$$

$$c_1 - \frac{(b_2 - b_3)^2}{a_1} = c_2 - \frac{(b_3 - b_1)^2}{a_2} = c_3 - \frac{(b_1 - b_2)^2}{a_3}.$$

For both the Clebsch and Steklov-Lyapunov cases there exists an additional quadratic integral.

State of the problem.

We consider the problem of description of pairs of functions

$$H = ap_1^2 + 2bp_1p_2 + cp_2^2 + dp_1 + ep_2 + f,$$

$$K = Ap_1^2 + 2Bp_1p_2 + Cp_2^2 + Dp_1 + Ep_2 + F,$$

that commute with respect to the standard Poisson bracket $\{p_i, q_j\} = \delta_{ij}$. Here $N = 2$ and the coefficients are functions of the variables q_1, q_2. This problem was considered by : Winternitz at all, Yehia, Ferapontov-Fordy, ...

The class of such Hamiltonians is invariant with respect to *canonical* transformations of the form

$$p_1 = k_1 \hat{p}_1 + k_2 \hat{p}_2 + k_3, \qquad p_2 = \bar{k}_1 \hat{p}_1 + \bar{k}_2 \hat{p}_2 + \bar{k}_3,$$

$$q_1 = \phi, \qquad q_2 = \bar{\phi},$$

where $k_i, \bar{k}_i, \phi, \bar{\phi}$ are some functions of $\hat{q}_1, \hat{q}_2$.

Using these transformations, we can reduce b and B to zero. After that we still have transformations with

$$q_1 \to \phi(q_1), \quad q_2 \to \bar{\phi}(q_2)$$

and shifts

$$p_1 \to p_1 + \frac{\partial F(q_1, q_2)}{\partial q_1}, \quad p_2 \to p_2 + \frac{\partial F(q_1, q_2)}{\partial q_2}.$$

Canonical form for the quasi-Stäckel Hamiltonians.

For the Hamiltonians of the form

$$H = ap_1^2 + cp_2^2 + dp_1 + ep_2 + f, \tag{1}$$

$$K = Ap_1^2 + Cp_2^2 + Dp_1 + Ep_2 + F \tag{2}$$

it follows from $\{H, K\} = 0$ that

$$a = \frac{S_1(q_1)}{\sigma_1(q_1) - \sigma_2(q_2)}, \quad c = \frac{S_2(q_2)}{\sigma_2(q_2) - \sigma_1(q_1)}$$

for some functions S_i, σ_i. If $\sigma_1' \neq 0$, $\sigma_2' \neq 0$, we may reduce σ_1 and σ_2 to q_1 and q_2.

Such a Hamiltonian H is called *quasi-Stäckel Hamiltonian.*

Theorem 1. Any pair H, K is equivalent to

$$H = \frac{U_1 - U_2}{q_1 - q_2}, \qquad K = \frac{q_2\, U_1 - q_1\, U_2}{q_1 - q_2}, \tag{3}$$

where

$$U_1 = S_1(q_1)\, p_1^2 + \frac{\sqrt{S_1(q_1) S_2(q_2)} Z_{q_1}}{(q_1 - q_2)}\, p_2 - \frac{S_1(q_1) Z_{q_1}^2}{4(q_1 - q_2)^2} + V_1(q_1, q_2),$$

$$U_2 = S_2(q_2)\, p_2^2 - \frac{\sqrt{S_1(q_1) S_2(q_2)} Z_{q_2}}{(q_1 - q_2)}\, p_1 - \frac{S_2(q_2) Z_{q_2}^2}{4(q_2 - q_1)^2} + V_2(q_1, q_2),$$

and

$$V_1 = \frac{1}{2}\sqrt{S_1(q_1)}\,\partial_{q_1}\left(\sqrt{S_1(q_1)}\frac{Z_{q_1}^2}{q_1-q_2}\right) + f_1(q_1),$$
$$V_2 = \frac{1}{2}\sqrt{S_2(q_2)}\,\partial_{q_2}\left(\sqrt{S_2(q_2)}\frac{Z_{q_2}^2}{q_2-q_1}\right) + f_2(q_2).$$

for some functions $Z(q_1, q_2)$, $S_i(q_i)$ and $f_i(q_i)$.

The Poisson bracket $\{H, K\}$ is equal to zero if and only if

$$\frac{\partial^2 Z}{\partial q_1 \partial q_2} = \frac{1}{2(q_2-q_1)}\left(\frac{\partial Z}{\partial q_1} - \frac{\partial Z}{\partial q_2}\right) \tag{4}$$

and

$$\left(Z_{q_1}\frac{\partial}{\partial q_2} - Z_{q_2}\frac{\partial}{\partial q_1}\right)\left(\frac{V_1 - V_2}{q_1 - q_2}\right) = 0. \tag{5}$$

The Stäckel Hamiltonians correspond to the trivial solution $Z = 0$. In this case

$$U_i = S_i(q_i)p_i^2 + f_i(q_i).$$

Here the variables are separated and the action function $\sigma(q_1, q_2)$ can be found in the form $\sigma = \sigma_1(q_1) + \sigma_2(q_2)$, where

$$S_i(q_i)(\sigma_i')^2 + f_i(q_i) - e_1 q_i - c = 0.$$

Consider the following non-trivial case.

Example 1. There exists the following solution of (4), (5):

$$Z(x,y) = x + y, \quad S_1(x) = S_2(x) = \sum_{i=0}^{6} c_i x^i,$$

$$f_1(x) = f_2(x) = -\frac{3}{4}c_6 x^4 - \frac{1}{2}c_5 x^3 + \sum_{i=0}^{2} k_i x^i,$$

where c_i, k_i are arbitrary constants.

It turns out that the Clebsch and the $so(4)$ Schottky-Manakov spinning tops are special cases of this model.

The Clebsch spinning top.

The Clebsch spinning top is defined by the Hamiltonian

$$H = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2) + \frac{1}{2}(\lambda_1\gamma_1^2 + \lambda_2\gamma_2^2 + \lambda_3\gamma_3^2)$$

which commutes with respect to the $e(3)$-Poisson brackets

$$\{M_i, M_j\} = i\,\varepsilon_{ijk}M_k, \qquad \{\gamma_i, \gamma_j\} = 0,$$

$$\{M_i, \gamma_j\} = i\,\varepsilon_{ijk}\gamma_k$$

with the first integral

$$K = (\lambda_1 M_1^2 + \lambda_2 M_2^2 + \lambda_3 M_3^2) -$$

$$\lambda_1\lambda_2\lambda_3\left(\frac{\gamma_1^2}{\lambda_1} + \frac{\gamma_2^2}{\lambda_2} + \frac{\gamma_3^2}{\lambda_3}\right).$$

Let us fix the values of the Casimir functions as follows

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = a^2, \qquad M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3 = l.$$

Using the parameterization

$$M_1 = \frac{1}{2}p_1(1 - q_1^2) + \frac{1}{2}p_2(1 - q_2^2) + \frac{l}{a}q_1,$$

$$M_2 = \frac{i}{2}p_1(1 + q_1^2) + \frac{i}{2}p_2(1 + q_2^2) - i\frac{l}{a}q_1,$$

$$M_3 = p_1q_1 + p_2q_2 - \frac{l}{a},$$

and

$$\gamma_1 = a\frac{1 - q_1q_2}{q_1 - q_2}, \quad \gamma_2 = ia\frac{1 + q_1q_2}{q_1 - q_2}, \quad \gamma_3 = a\frac{q_1 + q_2}{q_1 - q_2},$$

we can express H and K in terms of canonically conjugated variables $p_1, q_1,\ p_2, q_2$. As the result, we get the Hamiltonian H from Example 1 with

$$S(x) = 4(x - \lambda_1)(x - \lambda_2)(x - \lambda_3).$$

Classification.

To solve the classification problem, it suffices to investigate the compatibility conditions equations:

$$Z_{x,y} = \frac{Z_x - Z_y}{2(x-y)}$$

and

$$\left(Z_x \frac{\partial}{\partial y} - Z_y \frac{\partial}{\partial x}\right) \left(\frac{V_1 - V_2}{x-y}\right) = 0,$$

where

$$V_1 = \frac{1}{2}\sqrt{S_1(x)}\,\partial_x \left(\sqrt{S_1(x)}\frac{Z_x^2}{x-y}\right) + f_1(x),$$
$$V_2 = \frac{1}{2}\sqrt{S_2(y)}\,\partial_y \left(\sqrt{S_2(y)}\frac{Z_y^2}{y-x}\right) + f_2(y).$$

Here and below we use the notation

$$x = q_1, \qquad y = q_2.$$

Let us investigate the analytic behavior of solutions of the system (4), (5) at $x - y = 0$.

Lemma 1. The general solution of the Euler-Darboux equation (4) has the following decomposition:

$$Z(x,y) = A + \mathrm{Log}(x-y)\, B, \tag{6}$$

$$A = \sum_{0}^{\infty} a_i(x+y)\,(x-y)^{2i},$$

$$B = \sum_{0}^{\infty} b_i(x+y)\,(x-y)^{2i}.$$

In this formula a_0 and a_1 are arbitrary functions.

Substituting (6) into (5), we immediately obtain

Proposition 1. If series (6) satisfies (5), then $B = 0$.

Lemma 2. Any solution of the equations (4) with $B = 0$ is given by

$$Z(x,y) = z_0 + \delta\,(x+y) +$$

$$(x-y)^2 \sum_{k=0}^{\infty} \frac{g^{(2k)}(x+y)}{2^{(2k)} k!(k+1)!} (x-y)^{2k},$$

where $g(x)$ is arbitrary function and z_0, δ are arbitrary constants. Without loss of generality we put $z_0 = 0$. The parameter δ, plays a very important role in the classification of quasi-Stäckel Hamiltonians.

The function $g(x)$ is called *generating function* for $Z(x,y)$.

Let us describe in a close form all functions Z corresponding to **rational** generating functions g.

Taking $g(x) = x^n$, we obtain the infinite sequence of polynomial solutions Z_n for (4). In particular,

$$g(x) = 1 \iff Z_0(x, y) = (x - y)^2$$

$$g(x) = x \iff Z_1(x, y) = \frac{1}{2}(x + y)(x - y)^2.$$

This sequence can be constructed with the help of the "arising" operator

$$x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} - \frac{1}{2}(x + y)$$

acting on Z_0.

Moreover,

$$g_\mu(x) = \frac{1}{4}\frac{1}{x-2\mu} \iff$$

$$Z_\mu(x,y) = \sqrt{(\mu - x)(\mu - y)} + \frac{1}{2}(x+y).$$

The solution corresponding to the pole of order $n \geq 2$ can be obtained by differentiating of the latter formula with respect to μ.

Thus, we have constructed explicitly a solution Z with arbitrary rational generating function

$$g(x) = \sum_i c_i x^i + \sum_{i,j} d_{ij}(x-\mu_i)^{-j}.$$

Conjecture 1. For any integrable quasi-Stäckel Hamiltonian the generating function is *rational*.

Classification results.

Theorem 2 (non-symmetric case.) Suppose $S_1(x) \neq S_2(x)$, or $f_1(x) \neq f_2(x)$; then

$$g = \frac{1}{H}, \qquad S_{1,2} = W\,H \pm MH^{3/2},$$

$$f_{1,2} = -\frac{4W}{H} \mp 2MH^{-1/2} \pm aH^{1/2},$$

where g is the generating function of Z,

$$W(x) = w_3x^3 + w_2x^2 + w_1x + w_0,$$

$$H(x) = h_2x^2 + h_1x + h_0,$$

$$M(x) = m_2x^2 + m_1x + m_0.$$

Here w_i, h_i, m_i, a are arbitrary constants.

Consider now the symmetric case $S_1 = S_2, \quad f_1 = f_2$.

Theorem 3. Suppose $\delta = 0$. Then in the symmetric case the functions Z, $S = S_1$, $f = f_1$ satisfy (4), (5) iff

$$g = \frac{G}{S}, \qquad f = -\frac{4G^2}{S},$$

where

$$S(x) = s_5 x^5 + s_4 x^4 + s_3 x^3 + s_2 x^2 + s_1 x + s_0,$$

$$G(x) = g_3 x^3 + g_2 x^2 + g_1 x + g_0.$$

Here s_i, g_i are arbitrary constants.

All classification results have been obtained by substituting of the series

$$Z(x,y) = z_0 + \delta\,(x+y) +$$

$$(x-y)^2 \sum_{k=0}^{\infty} \frac{g^{(2k)}(x+y)}{2^{(2k)}k!(k+1)!}(x-y)^{2k}$$

to (5), equating the coefficients of different powers of $x-y$ and analyzing the overdetermined system of differential equations with respect to the functions g, S_i, f_i thus obtained.

Thus, to complete the classification of the quasi-Stäckel Hamiltonians, we must investigate the symmetric case with $\delta \neq 0$ (see, for instance, Example 1). Some examples of such Hamiltonians can be described as follows.

Generalized Manakov case.

The most general pair K, H of this kind can be written as

$$H = \frac{[h(y)U_1 - h(x)U_2] + h(x)a(y) - h(y)a(x)}{h(y)k(x) - h(x)k(y)},$$

$$K = \frac{[k(y)U_1 - k(x)U_2] + k(x)a(y) - k(y)a(x)}{h(y)k(x) - h(x)k(y)},$$

where $x = q_1$, $y = q_2$,

$$U_1 = S(x)p_1^2 + \delta\frac{\sqrt{S(x)S(y)}}{(x-y)}p_2 - \frac{\delta^2}{40}S''(x) +$$

$$\frac{\delta^2}{4}\frac{S'(x)}{(x-y)} - \frac{3\delta^2}{4}\frac{S(x)}{(x-y)^2},$$

U_2 can be obtained from U_1 by $x \leftrightarrow y, \quad p_1 \leftrightarrow p_2, \quad U_1 \leftrightarrow U_2$. Here $S(x)$ is an arbitrary sixth degree polynomial, δ is a parameter,

$$h(x) = h_2 x^2 + h_1 x + h_0, \qquad k(x) = k_2 x^2 + k_1 x + k_0,$$

$$a(x) = a_2 x^2 + a_1 x + a_0$$

are arbitrary polynomials such that $h(x) \neq const\, k(x)$. If $h(x) = 1$, $k(x) = x$ it coincides with Example 1.

For the above model the canonical forms correspond to

$$Z(x, y) = \sqrt{(x - \mu_1)(y - \mu_1)} + \sqrt{(x - \mu_2)(y - \mu_2)},$$

$$Z(x, y) = \sqrt{(x - \mu_1)(y - \mu_1)} + \frac{1}{2}(x + y),$$

$$Z(x, y) = x + y.$$

Deformed Steklov case.

A polynomial deformation of the Hamiltonian from Theorem 3 with respect to δ is given by

$$g = \frac{\tilde{G}}{S}, \qquad \tilde{G} = G - \frac{\delta}{10} S',$$

$$f = -\frac{4\tilde{G}^2}{S} - \frac{4\delta}{3}\tilde{G}' - \frac{\delta^2}{12} S'',$$

$$S(x) = s_5 x^5 + s_4 x^4 + s_3 x^3 + s_2 x^2 + s_1 x + s_0,$$

$$G(x) = g_3 x^3 + g_2 x^2 + g_1 x + g_0.$$

In the generic case

$$Z(x,\,y) = \sum_{i=1}^{5} \nu_i \sqrt{(\mu_i - x)(\mu_i - y)},$$

$$f(x) = -\frac{1}{16} \sum_{i=1}^{5} \frac{\nu_i^2 S'(\mu_i)}{x - \mu_i} + k_1 x + k_0,$$

where ν_i, μ_i, k_1, k_0 are arbitrary constants.

For the $so(4)$-Steklov case $s_5 = s_0 = 0,\ s_4, \ldots, s_1$ are arbitrary and

$$g(x) = \frac{4j_2}{x}, \qquad \delta = -\frac{j_1 + j_2}{2},$$

where j_i^2 are values of the Casimir functions.

Conjecture 2. Any quasi-Stäckel Hamiltonian with $\delta \neq 0$ belongs to one of the two classes described above.

The Hamilton-Jacobi equation
and separation of variables.

Consider the system of two stationary Hamilton-Jacobi equations

$$H\left(\frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \, q_1, q_2\right) = e_1$$

$$K\left(\frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \, q_1, q_2\right) = e_2.$$

Resolving this system w.r.t. first derivatives, we get a system of equations of the form

$$\frac{\partial S}{\partial q_i} = \Phi_i(q_1, q_2, e_1, e_2). \tag{7}$$

The system (7) is compatible i.e.

$$\frac{\partial \Phi_1}{\partial q_2} = \frac{\partial \Phi_2}{\partial q_1}.$$

The solution

$$S(q_1, q_2, e_1, e_2)$$

of system (7) is called the **action function**.

Differentiating S w.r.t. e_1, e_2, we obtain the coordinate functions $q_1(t)$, $q_2(t)$ of initial Hamiltonian system from

$$d\left(\frac{\partial S}{\partial e_1}\right) = d\,t, \qquad d\left(\frac{\partial S}{\partial e_2}\right) = 0.$$

The action function for quasi-Stäckel Hamiltonians.

Let

$$u = \frac{1}{2}\frac{Z_x}{x-y}\sqrt{\frac{y-\xi}{x-\xi}}, \qquad v = -\frac{1}{2}\frac{Z_y}{x-y}\sqrt{\frac{x-\xi}{y-\xi}},$$

where ξ is a parameter. If Z satisfies the Euler-Darboux equation, then $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$. Define $\sigma(x, y, \xi)$ as a solution of

$$\frac{\partial \sigma}{\partial x} = u, \qquad \frac{\partial \sigma}{\partial y} = v.$$

For example, in the generalized Steklov case $\sigma(x, y, \xi)$ equals to

$$-\frac{1}{2}\sum_{i=1}^{5} \nu_i \log \frac{\sqrt{x-\xi}\sqrt{y-\mu_i} + \sqrt{y-\xi}\sqrt{x-\mu_i}}{\sqrt{x-y}\sqrt{\mu_i-\xi}}.$$

Consider the expression

$$\Psi(x,y,\xi) = -e_2+e_1\xi+\frac{y-\xi}{x-y}\left(V_1(x,y)-\frac{S(x)Z_x^2}{4(x-\xi)(x-y)}\right)$$

$$-\frac{x-\xi}{x-y}\left(V_2(x,y)+\frac{S(y)Z_y^2}{4(y-\xi)(x-y)}\right).$$

Theorem. For any pair of quasi-Stäckel Hamiltonians the expression $\Psi(x,y,\xi)$ is a function of two variables ξ and $Y=\dfrac{\partial\sigma}{\partial\xi}$ only.

By this theorem, the equation $\Psi(x,y,\xi)=0$ defines an algebraic curve $\phi(\xi,Y)=0$. Let $\xi_k(x,y)$, $k=1,2,3$, be the roots of the qubic equation $\Psi(x,y,\xi)=0$.

Theorem. The action function S has the following form

$$S(x,y) = \sum_{k=1}^{3} \left[\sigma(x,y,\xi_k) - \int^{\xi_k} Y(\xi)\, d\xi \right],$$

where $Y(\xi)$ is the algebraic function on the curve

$$\phi(\xi, Y) = 0.$$

The action function for the Example 1.

Example 1. There exists the following solution of (4), (5):

$$Z(x,y) = x + y, \quad S_1(x) = S_2(x) = \sum_{i=0}^{6} c_i x^i,$$

$$f_1(x) = f_2(x) = -\frac{3}{4}c_6 x^4 - \frac{1}{2}c_5 x^3 + \sum_{i=0}^{2} k_i x^i,$$

where c_i, k_i are arbitrary constants.

In the case of Example 1 the curve is given by

$$\phi(\xi, Y) = 64c_6Y^6 + l(\xi)Y^4 + k(\xi)Y^2 - S(\xi) = 0,$$

where

$$k(\xi) = \frac{2}{5}S''(\xi) - \left(\frac{24c_4}{5} - 16k_2\right)\xi^2 + 16e_1\xi - 16e_2,$$

$$l(\xi) = \frac{2}{3}S^{IV}(\xi) - 2k''(\xi).$$

In the generic case this is a non-hyperelliptic curve of genus 4.

Let $P(\xi, \eta) = 0$ be an arbitrary cubic. Then $\phi(\xi, Y) = 0$ is a double cover over the cubic defined by $\eta = \xi^2 - 4Y^2$. The roots of S are branching points of the cover.

The system of equations

$$\phi(\xi, Y) = 0, \qquad Y^2 = \frac{1}{4}(\xi - x)(\xi - y)$$

is equivalent to a cubic equation with roots $\xi_i(x, y), \quad i = 1, 2, 3.$

Theorem. The action function is given by

$$S = \frac{1}{4}\sum_{n=1}^{3}\left[2\,\mathrm{arctanh}\,\frac{\xi_n - \frac{1}{2}(x+y)}{2\,Y(\xi_n)} - \int^{\xi_n}\frac{d\xi}{Y(\xi)}\right]$$

Differentiating the action with respect to the parameters, we get

$$dt = \sum_{n=1}^{3} \omega_1(\xi_n), \qquad 0 = \sum_{n=1}^{3} \omega_2(\xi_n).$$

Here ω_1, ω_2 belong to a basis

$$\omega_1(\xi) = \frac{d\xi}{Z}, \qquad \omega_2(\xi) = \frac{\xi\, d\xi}{Z},$$

$$\omega_3(\xi) = \frac{(\xi^2 - 4Y^2)d\xi}{Z}, \qquad \omega_4(\xi) = \frac{Y\, d\xi}{Z}$$

of holomorphic differentials on the curve $\phi(\xi, Y) = 0$. Here $Z = \frac{\partial\phi}{\partial Y}$.

The functions $Y(\xi_i)$ are linked by one constraint which can be rewritten in the variables (ξ, η) as

$$\eta_1(\xi_2 - \xi_3) + \eta_2(\xi_3 - \xi_1) + \eta_3(\xi_1 - \xi_2) = 0.$$

This means that the corresponding points (ξ_i, η_i) belongs to the intersection of the cubic and the straight line $\eta = \xi(x + y) - xy$.