Integrable dispersionless equations.

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The KP-hierarchy in the Sato form is defined by

$$L = D + a_1 D^{-1} + a_2 D^{-2} + \ldots$$

Consider the Lax equations

$$L_y = [A, L], \qquad L_t = [B, L], \tag{1}$$

where $\quad a_1 = u, \quad a_2 = u_x - v \quad$ and

$$A = \frac{1}{2}D^2 + u, \qquad B = \frac{1}{3}D^3 + uD + v.$$

Compatibility conditions for (1) implies

$$B_y - A_t = [A, B]$$

which is equivalent to

$$u_y = -\frac{1}{2}u_{xx} + v_x, \quad v_y - u_t = \frac{1}{2}v_{xx} - \frac{1}{3}u_{xxx} - uu_x.$$

The dispersionless KP-hierarchy (or the same the Ben-
ney chain) is defined by

$$L = p + \frac{a_1}{p} + \frac{a_2}{p^2} + \frac{a_3}{p^3} + \frac{a_4}{p^4} + ... \qquad (2$$

The dispersionless Lax equations is given by

$$L_y = \{A, L\}, \qquad L_t = \{B, L\},$$

where $\{f, g\} = f_p g_x - f_x g_p.$ Let

$$A = \frac{p^2}{2} + u, \qquad B = \frac{p^3}{3} + up + v;$$

then the corresponding dispersionless zero-curvatur
representation

$$B_y - A_t = \{A, B\}$$

is equivalent to $u_y = v_x, \quad v_y = u_t - uu_x.$

Notice that the relation $B_y - A_t = \{A, B\}$ can be re-garded as the compatibility condition $(\Phi_y)_t = (\Phi_t)$ for

$$\Phi_y = A(\Phi_x, y, t), \qquad \Phi_t = B(\Phi_x, y, t).$$

Thus the system

$$u_y = v_x, \quad v_y = u_t - uu_x$$

possesses the following pseudopotential representa-tion

$$\Phi_y = \frac{\Phi_x^2}{2} + u, \qquad \Phi_t = \frac{\Phi_x^3}{3} + u\Phi_x + v.$$

We consider systems of the form

$$\sum_{j=1}^{n} a_{ij}(\mathbf{u})\, u_{j,t} + \sum_{j=1}^{n} b_{ij}(\mathbf{u})\, u_{j,y} + \sum_{j=1}^{n} c_{ij}(\mathbf{u})\, u_{j,x} = 0,$$

where $i = 1, ..., n + k$. Here

$$\mathbf{u} = (u_1, ..., u_n)^t,$$

and $k = 0, 1, ...$ is called the *defect* of the system.

Equations of the form

$A_1 Z_{tt} + A_2 Z_{xt} + A_3 Z_{yt} + A_4 Z_{yy} + A_5 Z_{xy} + A_6 Z_{xx} =$

where $A_i = A_i(Z_x, Z_y, Z_t)$, correspond to $n = 3, k =$

Equations

$$F(Z_{tt}, Z_{xt}, Z_{yt}, Z_{yy}, Z_{xy}, Z_{xx}) = 0$$

correspond to $n = 5, k = 3$.

Definition. The system is called integrable if it pos-
sesses a pseudopotential representation

$$\Phi_y = A(\Phi_x, \mathbf{u}), \qquad \Phi_t = B(\Phi_x, \mathbf{u}).$$

Our goal now is to describe ”integrable” *pseudopo-*
tentials $A = \psi(p, \mathbf{u})$.

Consider first the simplest one-field case: $A = \psi(p, u$
The Benney hierarchy provides the following two ex-
amples

$$\psi = \frac{p^2}{2} + u, \quad \text{and} \quad \psi = \log(p - u).$$

One explicit example more:

$$\psi = \sqrt{u\,(p^2 + c_1) + c_2}.$$

Integrable pseudopotentials in the one-field cas

"Integrable" pseudopotentials $\psi(u,p)$ are given by

$$\psi_u = \frac{Q(\psi_p)}{\psi_{pp}}, \qquad \frac{\psi_{ppp}}{\psi_{pp}^2} = \frac{R(\psi_p)}{Q(\psi_p)}, \qquad (3$$

where R and Q are polynomials in ψ_p such tha
$\deg R \leq 3$, $\deg Q \leq 4$. In the generic case (3) implies

$$\frac{\psi_{ppp}}{\psi_{pp}^2} = \frac{k_1}{\psi_p - b_1(u)} + ... + \frac{k_4}{\psi_p - b_4(u)}, \qquad (4$$

$$b_i' = (1 - k_i)\, a \prod_{j \neq i} (b_i - b_j), \qquad i = 1, ..., 4, \qquad (5$$

where k_i are any constants such that $k_1 + ... + k_4 =$
and $b_i = b_i(u)$. The function $a(u)$ can be chose
arbitrarily due to the admissible transformations u
$s(u)$.

Let us choose

$$a = \frac{1}{(b_2 - b_3)(b_1 - b_4)} + \frac{1}{(b_1 - b_2)(b_3 - b_4)}.$$

Then the general solution of (5) is given by

$$b_1 = \frac{z_2 + u y_2}{z_1 + u y_1}, \quad b_2 = \frac{y_2}{y_1}, \quad b_3 = \frac{z_2 + y_2}{z_1 + y_1}, \quad b_4 = \frac{z_2}{z_1}$$

where $y_i(u)$ are two arbitrary solutions of the gype
geometric equation

$$u(u-1)\, y(u)'' + [(\alpha + \beta + 1)\, u - \gamma]\, y(u)' + \alpha\beta\, y(u) = 0$$

where $k_1 = 1 + \alpha - \gamma$, $k_2 = 1 - \alpha$, $k_3 = \gamma - \beta$, and

$$z_i = -u y_i + \frac{u(u-1)}{k_1 + k_2 + k_3 - 2}\, y_i'.$$

System (3) can be also integrated in quadratures.

Integrable 3D-systems related to the generalized hypergeometric functions

We construct new wide classes of pseudopotentia written in the following parametric form:

$$\Phi_y = F_1(\xi, \mathbf{u}), \qquad \Phi_t = F_2(\xi, \mathbf{u}), \qquad \Phi_x = F_3(\xi, \mathbf{u})$$

where $\mathbf{u} = (u_1, ..., u_n)$ and the ξ-dependence of func tions F_i is defined by the ODE

$$F_{i,\xi} = \phi_i(\xi, \mathbf{u}) \cdot \xi^{-s_1}(\xi - 1)^{-s_2}(\xi - u_1)^{-s_3}...(\xi - u_n)^{-s_{n+}}$$

Here $s_1, ..., s_{n+2}$ are arbitrary constants and ϕ_i ar some polynomials in ξ of degree $n - k$.

We call them *pseudopotentials of defect* k.

Consider the following system of linear PDEs:

$$\frac{\partial^2 h}{\partial u_j \partial u_k} = \frac{s_j}{u_j - u_k} \cdot \frac{\partial h}{\partial u_k} + \frac{s_k}{u_k - u_j} \cdot \frac{\partial h}{\partial u_j}, \quad i, j = 1, ..., n,$$

and

$$\frac{\partial^2 h}{\partial u_j \partial u_j} = -\left(1 + \sum_{k=1}^{n+2} s_k\right) \frac{s_j}{u_j(u_j - 1)} \cdot h +$$

$$\frac{s_j}{u_j(u_j - 1)} \sum_{k \neq j}^{n} \frac{u_k(u_k - 1)}{u_k - u_j} \cdot \frac{\partial h}{\partial u_k} +$$

$$\left(\sum_{k \neq j}^{n} \frac{s_k}{u_j - u_k} + \frac{s_j + s_{n+1}}{u_j} + \frac{s_j + s_{n+2}}{u_j - 1}\right) \cdot \frac{\partial h}{\partial u_j}$$

for unknown function $h(u_1, \ldots, u_n)$. If $n = 1$, then th
system coincides with the standard hypergeometr
equation

$$u(u-1)\, y(u)'' + [(\alpha+\beta+1)\, u - \gamma]\, y(u)' + \alpha\beta\, y(u) = 0$$

where $s_1 = -\alpha,\ s_2 = \alpha - \gamma,\ s_3 = \gamma - \beta - 1$.

Proposition 1. This system is compatible for an
constants $s_1, \ldots, s_{n+2}$. The dimension of the linea
space $\mathcal{H}$ of solutions of the system equals $n + 1$.

Pseudopotentials of defect 0

For any $g \in \mathcal{H}$. Put

$$S(g) = \sum_{1 \le i \le n} u_i(u_i - 1)(\xi - u_1)...\widehat{i}...(\xi - u_n)g_{u_i} +$$

$$(1 + \sum_{1 \le i \le n+2} s_i)(\xi - u_1)...(\xi - u_n)g$$

Example. In the case $n = 1$ we have

$$S(g, \xi) = u(u - 1)g_u + (1 + s_1 + s_2 + s_3)(\xi - u)g$$

where $u = u_1$.

Define function $P(g, \zeta)$ by

$$P(g, \zeta) = \int_0^{\zeta} S(g, \xi)(\xi - u_1)^{-s_1 - 1} ... (\xi - u_n)^{-s_n - 1} \times$$

$$\xi^{-s_{n+1} - 1}(\xi - 1)^{-s_{n+2} - 1} d\xi.$$

Let $g_0, g_1, g_2 \in \mathcal{H}$ be linear independent.

Theorem. The compatibility conditions $\Phi_{t_i t_j} = \Phi_{t_j}$
for the system

$$\Phi_{t_\alpha} = P(g_\alpha, \xi), \qquad \alpha = 0, 1, 2 \qquad (6$$

are equivalent to a system of PDEs for $u_1, ..., u_n$ o
the form:

$$\sum_{j=1}^{n} a_{ij}(\mathbf{u})\, u_{j,t_1} + \sum_{j=1}^{n} b_{ij}(\mathbf{u})\, u_{j,t_2} + \sum_{j=1}^{n} c_{ij}(\mathbf{u})\, u_{j,t_0} = 0$$

where $i = 1, ..., n$, and $t_0 = x$.

The explicit form of this system is given by

$$\sum_{i\neq j}((g_{1,u_j}g_{2,u_i}-g_{2,u_j}g_{1,u_i})\frac{u_j(u_j-1)u_{i,t_0}-u_i(u_i-1)u_j}{u_j-u_i}$$

$$(1+s_1+...+s_{n+2})(g_1g_{2,u_j}-g_2g_{1,u_j})u_{j,t_0}+$$

$$\sum_{i\neq j}((g_{2,u_j}g_{0,u_i}-g_{0,u_j}g_{2,u_i})\frac{u_j(u_j-1)u_{i,t_1}-u_i(u_i-1)u_j}{u_j-u_i}$$

$$(1+s_1+...+s_{n+2})(g_2g_{0,u_j}-g_0g_{2,u_j})u_{j,t_1}+$$

$$\sum_{i\neq j}((g_{0,u_j}g_{1,u_i}-g_{1,u_j}g_{0,u_i})\frac{u_j(u_j-1)u_{i,t_2}-u_i(u_i-1)u_j}{u_j-u_i}$$

$$(1+s_1+...+s_{n+2})(g_0g_{1,u_j}-g_1g_{0,u_j})u_{j,t_2}=0.$$

Pseudopotentials of defect $k > 0$

To define pseudopotentials of defect k, we fix k lin-early independent generalized hypergeometric func-tions $h_1, ..., h_k \in \mathcal{H}$. For any $g \in \mathcal{H}$ define $S_k(g, \xi$ by

$$S_k(g,\xi) = \frac{1}{\Delta} \sum_{1 \le i \le n-k+1} u_i(u_i - 1)(\xi - u_1) \times ...\hat{i}...$$

$$\times (\xi - u_{n-k+1}) \Delta_i(g).$$

Here

$$\Delta = \det \begin{pmatrix} h_1 & ... & h_k \\ h_{1,u_{n-k+2}} & ... & h_{k,u_{n-k+2}} \\ .... & ... & .... \\ h_{1,u_n} & ... & h_{k,u_n} \end{pmatrix},$$

$$\Delta_i(g) = \det \begin{pmatrix} g & h_1 & \dots & h_k \\ g_{u_i} & h_{1,u_i} & \dots & h_{k,u_i} \\ g_{u_{n-k+2}} & h_{1,u_{n-k+2}} & \dots & h_{k,u_{n-k+2}} \\ \dots\dots\dots & \dots & \dots & \dots\dots\dots \\ g_{u_n} & h_{1,u_n} & \dots & h_{k,u_n} \end{pmatrix} .$$

It is clear that $S_{n,k}(g,\xi)$ is a polynomial in ξ of degre $n-k$.

Example 3. In the simplest case $n = 2$, $k = 1$ w have

$$S_1(g,\xi) = u_1(u_1-1)(\xi-u_2)\frac{gh_{1,u_1} - g_{u_1}h_1}{h_1} +$$

$$u_2(u_2-1)(\xi-u_1)\frac{gh_{1,u_2} - g_{u_2}h_1}{h_1}.$$

Define the function $P_k(g,\xi)$ by

$$P_k(g,\xi) = \int_0^{\xi} S_k(g,\xi)(\xi-u_1)^{-s_1-1}...(\xi-u_{n-k+1})^{-s_{n-k+}$$

$$\times(\xi-u_{n-k+2})^{-s_{n-k+2}}...(\xi-u_n)^{-s_n}\xi^{-s_{n+1}-1}(\xi-1)^{-s_{n+2}$$

Theorem. The compatibility conditions $\Phi_{t_i t_j} = \Phi_{t_j$
for the system

$$\Phi_{t_\alpha} = P_k(g_\alpha,\xi), \qquad \alpha = 0,1,2 \qquad (7$$

are equivalent to the following system of PDEs fo
$u_1, ..., u_n$ of the defect k:

$$\sum_{1\le i\le n-k, i\neq j} (\Delta_j(g_q)\Delta_i(g_r) - \Delta_j(g_r)\Delta_i(g_q))$$

$$\times \frac{u_j(u_j-1)u_{i,t_s} - u_i(u_i-1)u_{j,t_s}}{u_j - u_i} +$$

$$\sum_{1\le i\le n-k, i\neq j} (\Delta_j(g_r)\Delta_i(g_s) - \Delta_j(g_s)\Delta_i(g_r))$$

$$\times \frac{u_j(u_j-1)u_{i,t_q} - u_i(u_i-1)u_{j,t_q}}{u_j - u_i} +$$

$$\sum_{1\le i\le n-k, i\neq j} (\Delta_j(g_s)\Delta_i(g_q) - \Delta_j(g_q)\Delta_i(g_s))$$

$$\times \frac{u_j(u_j-1)u_{i,t_r} - u_i(u_i-1)u_{j,t_r}}{u_j - u_i} = 0,$$

where $j = 1, ..., n-k$ and

$$\sum_{i=1}^{n-k+1} \Delta_i(g_r)u_{i,t_s} = \sum_{i=1}^{n-k+1} \Delta_i(g_s)u_{i,t_r},$$

$$\sum_{i=1}^{n-k+1} \Delta_i(g_r) \frac{u_m(u_m-1)u_{i,t_s} - u_i(u_i-1)u_{m,t_s}}{u_m - u_i} =$$

$$\sum_{i=1}^{n-k+1} \Delta_i(g_s) \frac{u_m(u_m-1)u_{i,t_r} - u_i(u_i-1)u_{m,t_r}}{u_m - u_i},$$

where $m = n-k+2, ..., n$. Here q, r, s run from 0 to
and $t_0 = x$.

Example 4. In the case $n = 3$, $k = 1$ the formula
can be rewritten as follows. Let h_1, g_0, g_1, g_2 be lin
early independent elements of $\mathcal{H}$. Denote by B_{ij} th
cofactors of the matrix

$$\begin{pmatrix} h_1 & g_0 & g_1 & g_2 \\ h_{1,u_1} & g_{0,u_1} & g_{1,u_1} & g_{1,u_1} \\ h_{1,u_2} & g_{0,u_2} & g_{1,u_2} & g_{1,u_1} \\ h_{1,u_3} & g_{0,u_3} & g_{1,u_3} & g_{1,u_3} \end{pmatrix}.$$

Define vector fields V_i by

$$V_1 = B_{22}\frac{\partial}{\partial t_0} + B_{23}\frac{\partial}{\partial t_1} + B_{24}\frac{\partial}{\partial t_2},$$

$$V_2 = B_{32}\frac{\partial}{\partial t_0} + B_{33}\frac{\partial}{\partial t_1} + B_{34}\frac{\partial}{\partial t_2},$$

$$V_3 = B_{42}\frac{\partial}{\partial t_0} + B_{43}\frac{\partial}{\partial t_1} + B_{44}\frac{\partial}{\partial t_2}.$$

Then the set of equations is equivalent to

$$V_1(u_2) = V_2(u_1), \quad V_2(u_3) = V_3(u_2), \quad V_3(u_1) = V_1($$

and

$$u_3(u_3 - 1)(u_1 - u_2)V_1(u_2) + u_1(u_1 - 1)(u_2 - u_3)V_2(u_3$$

$$+u_2(u_2 - 1)(u_3 - u_1)V_3(u_1) = 0.$$

There exist conservation laws of the form

$$\left(\frac{g_r}{h_1}\right)_{t_s} = \left(\frac{g_s}{h_1}\right)_{t_r}.$$

Introducing z such that $z_{t_r} = \frac{g_r}{h_1}$, we reduce the syster to a quasi-linear equation of the form

$$\sum_{i,j} P_{i,j}(z_{t_0}, z_{t_1}, z_{t_2})\, z_{t_i,t_j} = 0, \qquad i,j = 0,1,2. \qquad (8$$

In the paper by E. Feropontov an inexplicit descrip
tion of all integrable equations (8) was proposed. Th
equation constructed above corresponds to the gener
case in this classification. Indeed, it depends on fiv
essential parameters $s_1, ..., s_5$ which agrees with th
results of this paper.

Integrable elliptic pseudopotentials

If

$$\Phi_t = A(p, \mathbf{u}), \qquad \Phi_y = B(p, \mathbf{u}), \qquad \text{where} \quad p = \Phi_$$

is a pseudopotential representation for some integrab 3D-system, then for any $p \in \mathbb{C}$ the point $\left(\frac{A_{ppp}}{A_{pp}^2}, A_p\right.$ belongs to an algebraic curve of genus g, whose co efficients depend on $\mathbf{u}$.

Now we construct pseudopotentials and integrable sy tems related to the elliptic curve. For these system $\mathbf{u} = (u_1, \ldots, u_n, \tau)$, where τ is the parameter of th elliptic curve also being an unknown function in th system.

The coefficients of the systems are expressed in term
of the following elliptic generalization of hypergeo
metric functions in several variables:

$$g_{u_\alpha u_\beta} = s_\beta(\rho(u_\beta - u_\alpha) + \rho(u_\alpha + \eta) - \rho(u_\beta) - \rho(\eta))g_{u_\alpha} +$$

$$s_\alpha(\rho(u_\alpha - u_\beta) + \rho(u_\beta + \eta) - \rho(u_\alpha) - \rho(\eta))g_{u_\beta},$$

$$g_{u_\alpha u_\alpha} = s_\alpha \sum_{\beta\neq\alpha} (\rho(u_\alpha)+\rho(\eta)-\rho(u_\alpha-u_\beta)-\rho(u_\beta+\eta))g_{u_\beta}$$

$$(\sum_{\beta\neq\alpha} s_\beta\rho(u_\alpha - u_\beta) + (s_\alpha + 1)\rho(u_\alpha + \eta)+$$

$$s_\alpha\rho(-\eta) + (s_0 - s_\alpha - 1)\rho(u_\alpha) + 2\pi i r)g_{u_\alpha}-$$

$$s_0 s_\alpha(\rho'(u_\alpha) - \rho'(\eta))g,$$

$$g_\tau = \frac{1}{2\pi i}\sum_\beta (\rho(u_\beta + \eta) - \rho(\eta))g_{u_\beta} - \frac{s_0}{2\pi i}\rho'(\eta)g$$

for a single function $g(u_1, \ldots, u_n, \tau)$.

Here $\eta = s_1 u_1 + ... + s_n u_n + r\tau + \eta_0, \quad s_0 = -s_1 - ... - s$
where $s_1, ..., s_n, r, \eta_0$ are arbitrary constants, and

$$\theta(z) = \sum_{\alpha \in \mathbb{Z}} (-1)^\alpha e^{2\pi i(\alpha z + \frac{\alpha(\alpha-1)}{2}\tau)}, \qquad \rho(z) = \frac{\theta'(z)}{\theta(z)}.$$

We omit the second argument τ of the functions θ,
and use the notation

$$\rho'(z) = \frac{\partial \rho(z)}{\partial z}, \quad \rho_\tau(z) = \frac{\partial \rho(z)}{\partial \tau}, \quad \theta'(z) = \frac{\partial \theta(z)}{\partial z}, \quad \theta_\tau$$

It turns out that the dimension of the space of solu-
tions for the system equals $n+1$.

Describe pseudopotentials of defect $k = 0$ relate to the elliptic hypergeometric functions. The pseu dopotential $A_n(p, u_1, ..., u_n, \tau)$ is defined in a paramet ric form by

$$A_n = P_n(g_1, \xi), \qquad p = P_n(g_0, \xi),$$

where $g_1,\ g_0$ be linearly independent elliptic hyperge ometric functions

$$P_n(g, \xi) = \int_0^{\xi} S_n(g, \xi) e^{2\pi i r(\tau - \xi)} \times$$

$$\frac{\theta'(0)^{-s_1 - \dots - s_n} \theta(u_1)^{s_1} \dots \theta(u_n)^{s_n}}{\theta(\xi)^{-s_1 - \dots - s_n} \theta(\xi - u_1)^{s_1} \dots \theta(\xi - u_n)^{s_n}} d\xi,$$

and

$$S_n(g, \xi) = \sum_{1 \le \alpha \le n} \frac{\theta(u_\alpha)\theta(\xi - u_\alpha - \eta)}{\theta(u_\alpha + \eta)\theta(\xi - u_\alpha)} g_{u_\alpha} -$$

$$(s_1 + \dots + s_n) \frac{\theta'(0)\theta(\xi - \eta)}{\theta(\eta)\theta(\xi)} g.$$

We call them *elliptic pseudopotential of defect 0*.

Some important examples of pseudopotentials $A,$ related to the Whitham averaging procedure for inte-grable dispersion PDEs, to the Frobenious manifolds and to the WDVV-associativity equation were foun by B. Dubrovin and I. Krichever.

In the case $s_1 = ... = s_n = r = 0, \ \eta_0 \rightarrow 0$ ou pseudopotentials coincide with elliptic pseudopoten-tials constructed by Dubrovin and Krichever.