Calogero-Moser systems: A crossroads in mathematics and physics

1. What are Calogero-Moser systems?

General description: Integrable N-particle systems (on line or ring), characterized by interactions of elliptic, hyperbolic, trigonometric or rational type:

$$\begin{tikzcd} \text{2-period level} & & \mathrm{IV}\ (\text{ell.}) \arrow[dl] \arrow[dr] \arrow[dd] & \\ \text{1-period level} & \mathrm{III}\ (\text{trig.}) \arrow[dr] & & \mathrm{II}\ (\text{hyp.}) \arrow[dl] \\ \text{0-period level} & & \mathrm{I}\ (\text{rat.}) & \end{tikzcd}$$

Versions: classical/quantum, nonrelativistic/relativistic

1A. The nonrelativistic case

Simplest case: classical nonrelativistic rational CM:

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + g^2 \sum_{1\le j<k\le N} V(x_j - x_k), \quad V(x) = \frac{1}{x^2} \qquad (\mathrm{I})$$

To get II - IV, take $V(x)$ equal to

$$\nu^2/\sinh^2(\nu x), \quad \nu^2/\sin^2(\nu x), \quad \wp(x;\omega,\omega')$$

Reminder: For Hamiltonian $H(x,p)$ on phase space $\Omega \subset \mathbb{R}^{2N}$ with canonical symplectic form $\omega = \sum_{j=1}^{N} dx_j \wedge dp_j$, Hamilton's equations are given by the first order system

$$\dot{x}_j = \partial_{p_j} H, \quad \dot{p}_j = -\partial_{x_j} H, \quad j = 1, \dots, N$$

Also, the solution to this ODE system yields a 1-parameter flow e^{tH} of canonical transformations on Ω. Now $H(x,p)$ yields an integrable system if there exist N independent Hamiltonians $H_1, \dots, H_N$ (including H), whose flows commute. This can be expressed via

$$\{H_k, H_l\} = 0, \quad k, l = 1, \dots, N$$

with

$$\{F, G\}(x,p) \equiv \sum_{j=1}^{N} \left(\partial_{x_j} F \, \partial_{p_j} G - \partial_{p_j} F \, \partial_{x_j} G \right) \quad \text{(Poisson bracket)}$$

For above H, Poisson commuting Hamiltonians are given by

$$H_1 = \sum_{j=1}^{N} p_j, \quad H_2 = H, \quad H_k = \frac{1}{k} \sum_{j=1}^{N} p_j^k + \text{lower order in } p_j, \quad k = 3, \dots, N$$

Quantization: take $p_j \to -i\hbar \partial_{x_j} \equiv \hat{p}_j$ ($\hbar$ = Planck's c^{st})

With suitable ordering for $k > 2$, get N commuting PDOs.

1B. The relativistic case

Let $c>0$ be speed of light. Set $\beta = 1/c$ and introduce

$$H = \frac{M}{\beta^2} \sum_{j=1}^{N} \cosh\left(\beta \frac{p_j}{M}\right) \prod_{k\neq j} f(x_j - x_k)$$

$$P = \frac{M}{\beta} \sum_{j=1}^{N} \sinh\left(\beta \frac{p_j}{M}\right) \prod_{k\neq j} f(x_j - x_k)$$

Choosing $f^2(x) = a + b\,\wp(x)$ implies

$$\{H, P\} = 0 \qquad (\text{translation invariance})$$

Clearly,

$$\{H, B\} = P \qquad B = -M \sum_{j=1}^{N} x_j \quad (\text{Lorentz boost})$$

$$\{P, B\} = \beta^2 H$$

and taking $a=1$, $b = g^2\beta^2/M^2$ ensures

$$\lim_{\beta\to 0} \left(H - N\frac{M}{\beta^2}\right) = \frac{1}{2M} \sum_{j=1}^{N} p_j^2 + \frac{g^2}{M} \sum_{j<k} \wp(x_j - x_k) = H_{nr}$$

$$\lim_{\beta\to 0} P = \sum_{j=1}^{N} p_j = P_{nr}$$

$\therefore$ Get relativistic version of nonrelativistic CM systems

As a bonus, get Poisson commuting Hamiltonians

$$S_{\pm \ell}(x,p) = \sum_{\substack{J \subset \{1,\dots,N\} \\ |J| = \ell}} \exp\left(\pm\beta \sum_{j\in J} \frac{p_j}{M}\right) \prod_{\substack{j\in J \\ k \notin J}} f(x_j - x_k), \quad \ell = 1,\dots,N$$

Quantization. • Should interpret

$$\exp(\beta \hat{p}_j/M) = \exp(-i \tfrac{\hbar}{Mc} \partial_{x_j})$$

as translation, i.e.,

$$(T_j \Psi)(x_1, \dots, x_j, \dots, x_N) = \Psi(x_1, \dots, x_j - i\tfrac{\hbar}{Mc}, \dots, x_N)$$

Hence, the quantum Hamiltonians $\hat{S}_{\pm l}(x, -i\hbar\nabla_x)$ are analytic difference operators (AΔOs).

• Need to find integrable quantization, i.e, ordering such that the Hamiltonians commute:

$$[\hat{S}_{\pm l}, \hat{S}_{\pm m}] = 0, \qquad l, m = 1, \dots, N$$

• Solution: Using suitable factorization $f(x) = f_-(x) f_+(x)$, get commuting AΔOs

$$\hat{S}_{\pm l} = \sum_{|J|=l} \prod_{\substack{j\in J \\ k\notin J}} f_{\mp}(x_j - x_k) \cdot \prod_{j\in J} \exp(\pm\beta \hat{p}_j/M) \cdot \prod_{\substack{j\in J \\ k\notin J}} f_{\pm}(x_j - x_k)$$

• For $f^2(x) = 1 + \dfrac{\sin^2\tau}{\sinh^2 \nu x}$, should take $f_{\pm}^2(x) = \dfrac{\sinh(\nu x \pm i\tau)}{\sinh(\nu x)}$

• Should take sinh → Weierstraß σ-function in $f_{\pm}(x)$ to get commuting AΔOs of type IV.

1C. Generalizations

- Analytic continuation in x_j yields systems with two 'charges' ($1/\sinh^2 y \to -1/\cosh^2 y$, repulsive $\to$ attractive)
- Above CM correspond to A_{N-1} root system; there also exist CM versions for $B_N, C_N, D_N, BC_N, E_6, E_7, E_8, F_4, G_2$, and for the super Lie algebra root systems
- Versions with internal degrees of freedom ('spins') exist
- Limits of above yield other integrable systems:
  - various external field couplings in type I - III
  - spin chains (Haldane/Shastry, Inozemtsev)
  - Toda systems
  - delta function boson gas

2. Relations with other areas

Preamble. CM systems have ties with a great many subfields in physics and maths. Often, this involves the key objects encapsulating the explicit solution to the joint Hamilton/Schrödinger equations on the classical/quantum level, namely, the action-angle map/joint eigenfunction transform, respectively.

A (non-exhaustive) list now follows, roughly in order of

pure maths $\rightarrow$ applied maths $\rightarrow$ physics

- symplectic geometry (moment map, Marsden-Weinstein reduction, action-angle theory)
- algebraic geometry (Riemann surfaces, Jacobian varieties, theta functions, Baker-Akhiezer functions)
- Lie groups and symmetric spaces, Lie algebras and root systems, representation theory, and 'quantum' versions of all these ($q \rightarrow 1$ corresponds to $c \rightarrow \infty$)
- combinatorics (as related to polynomials of Askey-Wilson, Hall-Littlewood, Macdonald, Koornwinder type)
- special functions of Heun, Lamé, hypergeometric type, and 'relativistic' analogs; multi-variate versions thereof, and generalized gamma functions
- theory of analytic difference equations (Schrödinger equation for AΔOs)
- Nevanlinna theory
- Hilbert space issues (eigenfunction expansion theory, self-adjointness/isometry questions, scattering theory)

- classical soliton theory (the soliton solutions of many nonlinear 2D evolution equations can be obtained from the relativistic hyperbolic CM systems)
- quantum soliton theory (particle number and momenta conserved, scattering factorizes)
- solvable models in statistical mechanics (6- and 8-vertex, XXZ and XYZ, Potts models)
- random matrix theory (special couplings in CM)
- 2D Yang-Mills (on the circle)
- 4D supersymmetric gauge field models (Seiberg-Witten theory)
- quantum chaos (level repulsion)

Moreover, CM systems have connections with:

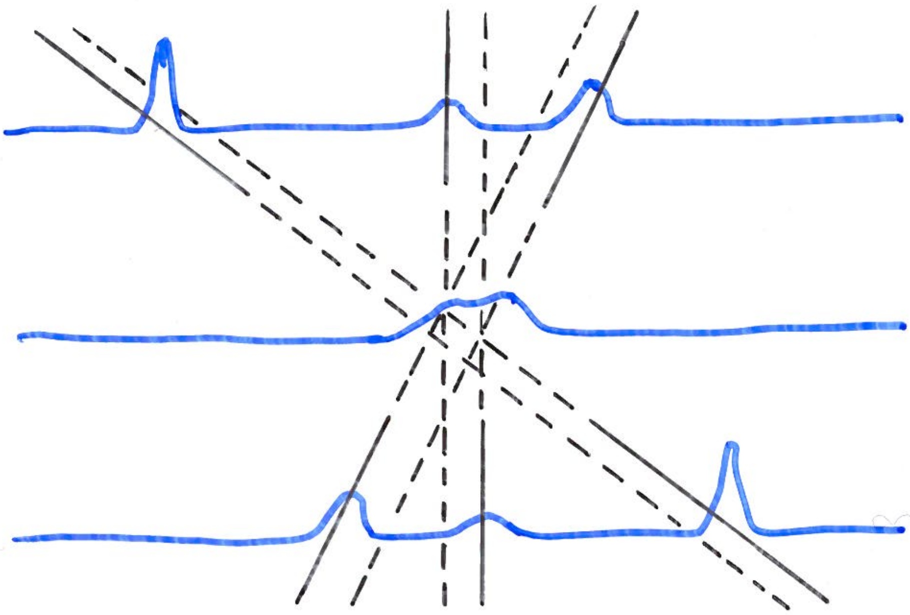
- Sklyanin, affine Hecke, Kac-Moody, Virasoro, W-algebras
- Painlevé, Knizhnik-Zamolodchikov, Yang-Baxter, WDVV eqs.
- Hitchin, Gaudin, WZNW and matrix models
- operaters of Dunkl, Cherednik and Polychronakos type
- Huygens' principle
- bispectral problem

3. The relation to the sine-Gordon solitons

Consider N-soliton solution to the sine-Gordon eq.

$$\varphi'' - \ddot{\varphi} = \sin(\varphi).$$

e.g. for $N=3$:



$\varphi'(T,y)$

$\varphi'(0,y)$

$\varphi'(-T,y)$

Characteristic features, preserved under quantization:

$$\left.\begin{array}{l}\text{conservation of momenta}\\ \text{factorization of phase shift}\end{array}\right\}\ \underline{\text{soliton scattering}}$$

<u>Fact</u>: The $\tau = \pi/2$ hyperbolic relativistic CM system yields the same soliton scattering on the classical level

<u>Conjecture</u>: This remains true on the quantum level.

(This is proved for $N=2$.)

Specifically, using the Poisson commuting space-time translation generators

$$H = \sum_{j=1}^{N} \cosh(p_j) \prod_{k\neq j} \coth(x_j - x_k)/2 ,$$

$$P = \sum_{j=1}^{N} \sinh(p_j) \prod_{k\neq j} \coth(x_j - x_k)/2 ,$$

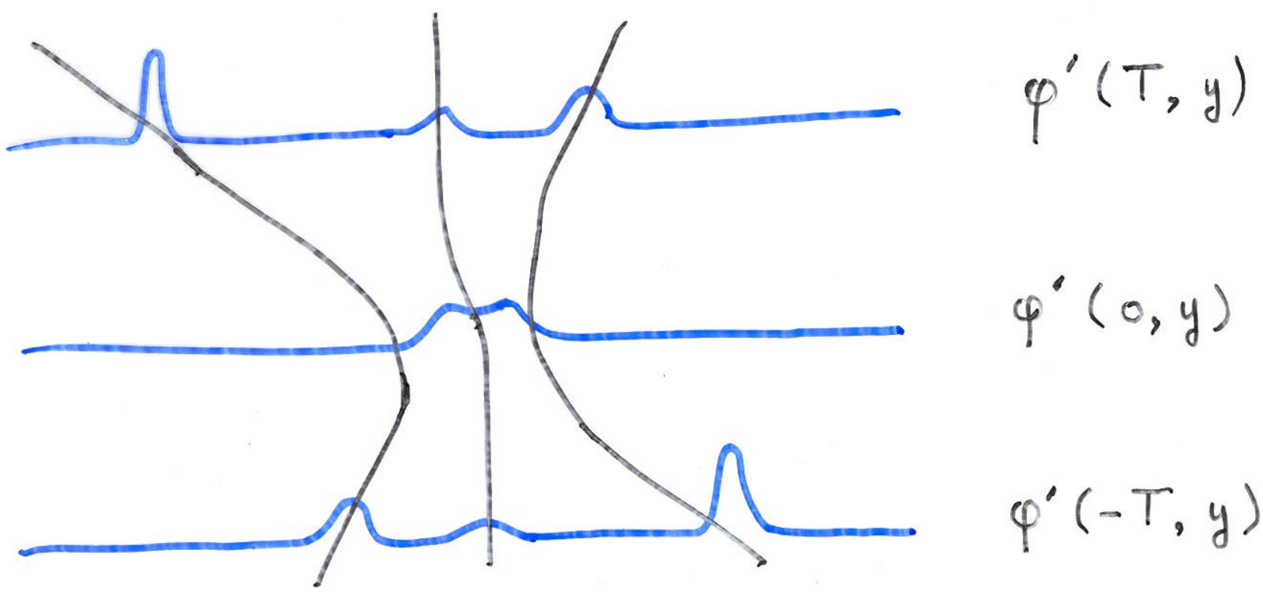
define the space-time dependent generalized positions

$$x_j(t,y) = \left(e^{tH - yP}(x,p)\right)_j , \quad j = 1,\dots,N$$

Then the function

$$\varphi(t,y) = 4 \sum_{j=1}^{N} \mathrm{Arctan}\left(e^{x_j(t,y)}\right)$$

is an N-soliton solution to $\varphi'' - \ddot{\varphi} = \sin\varphi$. Requiring $x_j(t,y) = 0$ yields soliton space-time trajectories $y_j(t)$:



- $s s$ repel, but $s \bar{s}$ attract ($\coth \to \tanh$)
- $N = 2$ quantum correspondence involves 'relativistic' hypergeometric function

4. The 'relativistic' hypergeometric function

4A. Some ${}_2F_1$-reminders

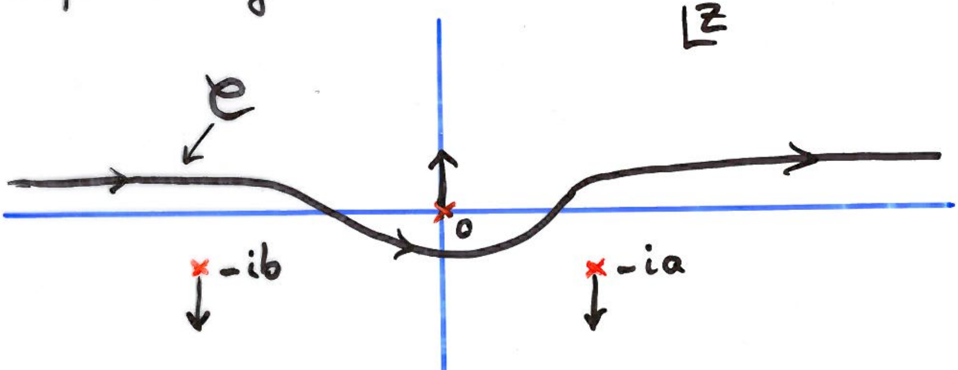
Gauss series for the hypergeometric function

$$ {}_2F_1(a,b,c;w) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \cdot \frac{\Gamma(b+n)}{\Gamma(b)} \cdot \frac{\Gamma(c)}{\Gamma(c+n)} \cdot \frac{w^n}{n!}, \qquad |w|<1 $$

Analytic continuation to $|\mathrm{Arg}(-w)|<\pi$ via Barnes representation

$$ \int_{\mathcal{C}} dz\,(-w)^{-iz} \cdot \frac{\Gamma(iz)\Gamma(c)}{2\pi\Gamma(c-iz)} \cdot \frac{\Gamma(a-iz)\Gamma(b-iz)}{\Gamma(a)\Gamma(b)} $$

with $\mathcal{C}$ defined by



Putting

$$ \psi(\nu,g,\tilde{g};x,p) \equiv {}_2F_1\left(\tfrac{1}{2}\left(g+\tilde{g}+\tfrac{ip}{\nu}\right), \tfrac{1}{2}\left(g+\tilde{g}-\tfrac{ip}{\nu}\right), g+\tfrac{1}{2}; -\mathrm{sh}^2\nu x\right) $$

yields solution to Schrödinger equation

$$ H\psi = \left(p^2+\nu^2(g+\tilde{g})^2\right)\psi, \qquad H \equiv -\partial_x^2 - 2\nu\left[g\,\mathrm{cth}(\nu x)+\tilde{g}\,\mathrm{th}(\nu x)\right]\partial_x $$

Need weight function similarity to get Calogero-Moser (BC_1) form $-\partial_x^2 + \nu^2 g(g-1)/\mathrm{sh}^2\nu x - \nu^2\tilde{g}(\tilde{g}-1)/\mathrm{ch}^2\nu x + c^{st}$

4B. The hyperbolic gamma function

Fix $a_+, a_- > 0$, put $a \equiv (a_+ + a_-)/2$. Define hyperbolic G-fnc. by

$$G(a_+,a_-;z) = \exp\left[i \int_0^\infty \frac{dy}{y} \left(\frac{\sin 2yz}{2\,\mathrm{sh}(a_+y)\,\mathrm{sh}(a_-y)} - \frac{z}{a_+a_-y} \right) \right], \quad |\mathrm{Im}\, z| < a$$

Pertinent properties

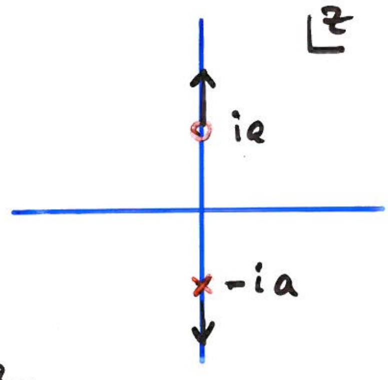
— G is meromorphic solution to AΔEs (analytic difference eqs.)

$$\frac{G(z + i a_\delta/2)}{G(z - i a_\delta/2)} = 2\,\mathrm{ch}\left(\frac{\pi}{a_{-\delta}} z\right), \quad \delta = +,-$$

— Clearly, $G(-z) = 1/G(z)$, $G(a_-,a_+;z) = G(a_+,a_-;z)$,

$$G(\lambda a_+, \lambda a_-; \lambda z) = G(a_+,a_-;z)$$

— Zeros and poles of G given by



$$\left.\begin{matrix}\text{zeros}\\ \text{poles}\end{matrix}\right\} = \pm i\,[a + k a_+ + l a_-], \quad k, l \in \mathbb{N}$$

Pole at $z = -ia$ simple with residue $\frac{i}{2\pi}\sqrt{a_+a_-}$

— Letting $g \equiv -i\,\mathrm{Ln}\, G$, $\varepsilon > 0$, $a_\ell \equiv \max(a_+,a_-)$, one has

$$\pm g(a_+,a_-;z) = -\frac{\pi z^2}{2a_+a_-} - \frac{\pi}{24}\left(\frac{a_+}{a_-} + \frac{a_-}{a_+}\right) + O\left(\exp\left[\pm\left(\varepsilon - 2\pi/a_\ell\right) z\right]\right), \quad \mathrm{Re}\, z \to \pm\infty$$

4C. The R-function

Fix 'coupling constants' $c \in (0,\infty)^4$ such that $s_j < a$, with

$s_1 \equiv c_0 + c_1 - \frac{a_-}{2}$, $s_2 \equiv c_0 + c_2 - \frac{a_+}{2}$, $s_3 \equiv c_0 + c_3$. Then set

$$R(a_+, a_-, c; v, \hat{v}) = \frac{1}{\sqrt{a_+ a_-}} \int_{\mathcal{C}} dz\, I(a_+, a_-, c; v, \hat{v}, z),$$

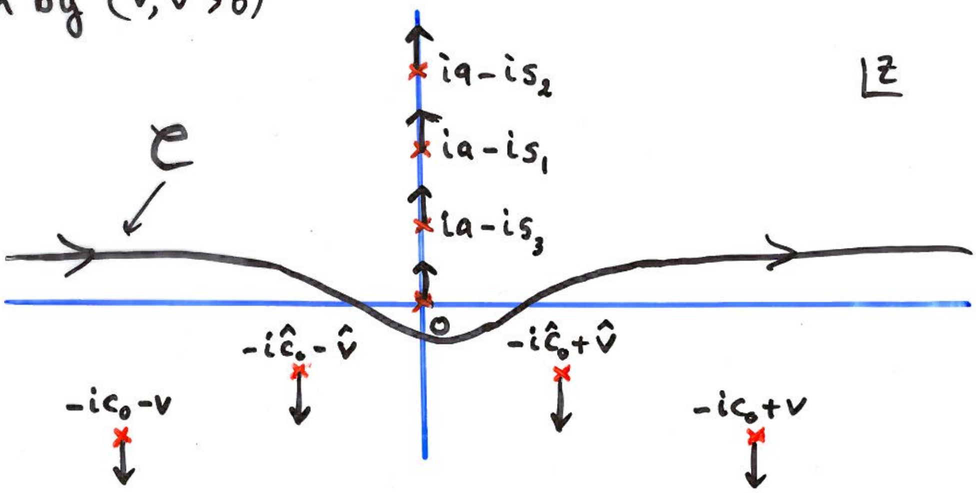
with

$$I \equiv F(c_0; v, z) K(a_+, a_-, c; z) F(\hat{c}_0; \hat{v}, z), \qquad \hat{c}_0 \equiv \tfrac{1}{2}(c_0 + c_1 + c_2 + c_3),$$

$$F(d; y, z) \equiv \left(\frac{G(z + y + id - ia)}{(z = 0)} \right) (y \to -y),$$

$$K(a_+, a_-, c; z) \equiv \frac{1}{G(z + ia)} \cdot \prod_{j=1}^{3} \frac{G(is_j)}{G(z + is_j)},$$

and $\mathcal{C}$ given by $(v, \hat{v} > 0)$



From G-asymptotics get

$$I(z) = O\left(\exp\left[\mp 2\pi z \left(\frac{1}{a_+} + \frac{1}{a_-}\right)\right]\right), \quad \mathrm{Re}\, z \to \pm\infty$$

$\therefore$ R well defined, meromorphic in $v, \hat{v}$, analytic for $\mathrm{Re}\, v, \mathrm{Re}\, \hat{v} \neq 0$.

4D. The hyperbolic Askey-Wilson A∆Os

Defining

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$\hat{c} \equiv Jc \qquad (\Rightarrow c_0 + c_j = \hat{c}_0 + \hat{c}_j, \; s_j = \hat{s}_j, \; j = 1,2,3),$$

note symmetry properties

$$R(a_+, a_-, c; v, \hat{v}) = R(a_+, a_-, \hat{c}; \hat{v}, v)$$

$$R(a_+, a_-, c; v, \hat{v}) = R(a_-, a_+, Ic; v, \hat{v})$$

Now put

$$s_\delta(z) \equiv \mathrm{sh}\left(\frac{\pi}{a_\delta} z\right), \quad c_\delta(z) \equiv \mathrm{ch}\left(\frac{\pi}{a_\delta} z\right), \quad \delta = +,-$$

and introduce A∆Os (analytic difference operators)

$$A_\delta(c; y) = C_\delta(y)\left(T_{ia_{-\delta}} - 1\right) + C_\delta(-y)\left(T_{-ia_{-\delta}} - 1\right) + 2c_\delta(2i\hat{c}_0)$$

with

$$C_\delta(y) \equiv \frac{s_\delta(y - ic_0)}{s_\delta(y)} \cdot \frac{c_\delta(y - ic_1)}{c_\delta(y)} \cdot \frac{s_\delta(y - ic_2 - ia_{-\delta}/2)}{s_\delta(y - ia_{-\delta}/2)} \cdot \frac{c_\delta(y - ic_3 - ia_{-\delta}/2)}{c_\delta(y - ia_{-\delta}/2)}$$

$$(T_\alpha F)(y) \equiv F(y - \alpha), \quad \alpha \in \mathbb{C}$$

Fact. R is joint eigenfunc. of $A_+(c; v)$, $A_-(Ic; v)$, $A_+(\hat{c}; \hat{v})$, $A_-(I\hat{c}; \hat{v})$

with eigenvalues $\quad 2c_+(2\hat{v}),\; 2c_-(2\hat{v}),\; 2c_+(2v),\; 2c_-(2v)$.

4E. Further R-features

- The specialization

$$R(c; v, i\hat{c}_0 + i n a_-) = P_n(c_+(2v)), \quad n \in \mathbb{N}$$

yields polynomials $P_n(x)$ of degree n; these are the Askey-Wilson polynomials.

- A reparametrized and weight-function similarity-transformed version $\mathcal{E}(\gamma; v, \hat{v})$ of $R(c; v, \hat{v})$ has D_4-symmetry in the parameters $\gamma_0, \ldots, \gamma_3$.

- This function has plane-wave asymptotics

$$\mathcal{E}(\gamma; v, \hat{v}) \sim \exp(2\pi i v\hat{v}/a_+a_-) + s(\gamma; \hat{v})\exp(-2\pi i v\hat{v}/a_+a_-), \quad v \to \infty,$$

with s a phase; it yields a generalized cosine transform on $L^2([0,\infty), dv)$ that is unitary for γ in a polytope.

- In a larger polytope a finite number of bound states occurs, yielding the DHN-spectrum upon specializing γ to its sine-Gordon values.

- The R-function can be tied in with Faddeev's notion of modular quantum group (F. v. d. Bult).