Solvability of the E_8 trigonometric system

Juan Carlos López Vieyra*
Instituto de Ciencias Nucleares, UNAM

Recent Developments in Integrable Systems and their Transition to Chaos
Symposium in Honor of
Francesco Calogero
on the Occasion of his 75th Birthday

*Collaboration with A. Turbiner, M. García (ICN-UNAM) and K.G. Boreskov (ITEP)

Olshanetsky-Perelomov Hamiltonians

Olshanetsky-Perelomov Hamiltonians

- 80's: Discovery of a remarkable family of quantum Hamiltonians associated to the root systems of the classical (A_n (Calogero-Sutherland) , B_n, C_n, D_n) and exceptional (G_2, F_4, $E_{6,7,8}$) Lie algebras, and non-crystalographic ($H_{3,4}$, $I_2(k)$) root systems

Olshanetsky-Perelomov Hamiltonians

- 80's: Discovery of a remarkable family of quantum Hamiltonians associated to the root systems of the classical (A_n (Calogero-Sutherland) , B_n, C_n, D_n) and exceptional (G_2, F_4, $E_{6,7,8}$) Lie algebras, and non-crystalographic ($H_{3,4}$, $I_2(k)$) root systems

- They have the property of **complete integrability** and **exact solvability**

Olshanetsky-Perelomov Hamiltonians

- 80's: Discovery of a remarkable family of quantum Hamiltonians associated to the root systems of the classical (A_n (Calogero-Sutherland) , B_n, C_n, D_n) and exceptional (G_2, F_4, $E_{6,7,8}$) Lie algebras, and non-crystalographic ($H_{3,4}$, $I_2(k)$) root systems

- They have the property of **complete integrability** and **exact solvability**

- So far, any quantum O-P Hamiltonian, written in certain variables, has the form of a second order differential operator with polynomial coefficients **(algebraic form)**
  The goal is to check it for E_8-trigonometric model

Olshanetsky-Perelomov Hamiltonians

- 80's: Discovery of a remarkable family of quantum Hamiltonians associated to the root systems of the classical (A_n (Calogero-Sutherland) , B_n, C_n, D_n) and exceptional (G_2, F_4, $E_{6,7,8}$) Lie algebras, and non-crystalographic ($H_{3,4}$, $I_2(k)$) root systems

- They have the property of **complete integrability** and **exact solvability**

- So far, any quantum O-P Hamiltonian, written in certain variables, has the form of a second order differential operator with polynomial coefficients **(algebraic form)**
  The goal is to check it for E_8-trigonometric model

- The eigenfunctions of the (algebraic) Hamiltonian are always polynomials

Olshanetsky-Perelomov Hamiltonians

- 80's: Discovery of a remarkable family of quantum Hamiltonians associated to the root systems of the classical (A_n (Calogero-Sutherland) , B_n, C_n, D_n) and exceptional (G_2, F_4, $E_{6,7,8}$) Lie algebras, and non-crystalographic ($H_{3,4}$, $I_2(k)$) root systems
- They have the property of **complete integrability** and **exact solvability**
- So far, any quantum O-P Hamiltonian, written in certain variables, has the form of a second order differential operator with polynomial coefficients **(algebraic form)**
  The goal is to check it for E_8-trigonometric model
- The eigenfunctions of the (algebraic) Hamiltonian are always polynomials
- The eigenvalue is a second-degree polynomial in the *quantum numbers*

$$\mathcal{H}_{E_8} = -\frac{1}{2}\Delta^{(8)} + \frac{g\beta^2}{4}\sum_{j<i=1}^{8}\left[\frac{1}{\sin^2\frac{\beta}{2}(x_i+x_j)} + \frac{1}{\sin^2\frac{\beta}{2}(x_i-x_j)}\right]$$

$$+\frac{g\beta^2}{4}\sum_{\{\nu_j\}}\frac{1}{\left[\sin^2\frac{\beta}{4}\left(x_8+\sum_{j=1}^{7}(-1)^{\nu_j}x_j\right)\right]}\ , \ \nu_j = 0, 1,\ \sum_{j=1}^{7}\nu_j \text{ is even}$$

E_8 trig Hamiltonian in 8-dimensional Euclidean space with coords $x_1, x_2, \ldots x_8$

$$\mathcal{H}_{E_8} = -\frac{1}{2}\Delta^{(8)} + \frac{g\beta^2}{4}\sum_{j<i=1}^{8}\left[\frac{1}{\sin^2\frac{\beta}{2}(x_i+x_j)} + \frac{1}{\sin^2\frac{\beta}{2}(x_i-x_j)}\right]$$

$$+\frac{g\beta^2}{4}\sum_{\{\nu_j\}}\frac{1}{\left[\sin^2\frac{\beta}{4}\left(x_8+\sum_{j=1}^{7}(-1)^{\nu_j}x_j\right)\right]}\,,\ \nu_j = 0, 1,\ \sum_{j=1}^{7}\nu_j \text{ is even}$$

- $g = \nu(\nu-1) > -\frac{1}{4}$ **(coupl const.)** $\beta \in \mathbb{R}$, **param** $\propto$ **(period)**$^{-1}$.

E_8 trig Hamiltonian in 8-dimensional Euclidean space with coords $x_1, x_2, \dots x_8$

$$\mathcal{H}_{E_8} = -\frac{1}{2}\Delta^{(8)} + \frac{g\beta^2}{4}\sum_{j<i=1}^{8}\left[\frac{1}{\sin^2\frac{\beta}{2}(x_i+x_j)} + \frac{1}{\sin^2\frac{\beta}{2}(x_i-x_j)}\right]$$

$$+\frac{g\beta^2}{4}\sum_{\{\nu_j\}}\frac{1}{\left[\sin^2\frac{\beta}{4}\left(x_8+\sum_{j=1}^{7}(-1)^{\nu_j}x_j\right)\right]}\ ,\ \nu_j = 0, 1,\ \sum_{j=1}^{7}\nu_j \text{ is even}$$

- $g = \nu(\nu-1) > -\frac{1}{4}$ **(coupl const.)** $\beta \in \mathbb{R}$, **param** $\propto$ **(period)**$^{-1}$.
- Symmetry: E_8 Weyl group (order $= 696729600$)

E_8 trig Hamiltonian in 8-dimensional Euclidean space with coords $x_1, x_2, \ldots x_8$

$$\mathcal{H}_{E_8} = -\frac{1}{2}\Delta^{(8)} + \frac{g\beta^2}{4}\sum_{j<i=1}^{8}\left[\frac{1}{\sin^2\frac{\beta}{2}(x_i+x_j)} + \frac{1}{\sin^2\frac{\beta}{2}(x_i-x_j)}\right]$$

$$+\frac{g\beta^2}{4}\sum_{\{\nu_j\}}\frac{1}{\left[\sin^2\frac{\beta}{4}\left(x_8+\sum_{j=1}^{7}(-1)^{\nu_j}x_j\right)\right]}\ , \ \nu_j = 0, 1, \ \sum_{j=1}^{7}\nu_j \text{ is even}$$

- $g = \nu(\nu-1) > -\frac{1}{4}$ **(coupl const.)** $\beta \in \mathbb{R}$, **param** $\propto$ **(period)**$^{-1}$.
- Symmetry: E_8 Weyl group (order $= 696729600$)
- The configuration space is the principal E_8 Weyl alcove.

E_8 trig Hamiltonian in 8-dimensional Euclidean space with coords $x_1, x_2, \ldots x_8$

$$\mathcal{H}_{E_8} = -\frac{1}{2}\Delta^{(8)} + \frac{g\beta^2}{4}\sum_{j<i=1}^{8}\left[\frac{1}{\sin^2\frac{\beta}{2}(x_i+x_j)} + \frac{1}{\sin^2\frac{\beta}{2}(x_i-x_j)}\right]$$

$$+\frac{g\beta^2}{4}\sum_{\{\nu_j\}}\frac{1}{\left[\sin^2\frac{\beta}{4}\left(x_8+\sum_{j=1}^{7}(-1)^{\nu_j}x_j\right)\right]}\ ,\ \nu_j = 0, 1,\ \sum_{j=1}^{7}\nu_j \text{ is even}$$

- $g = \nu(\nu-1) > -\frac{1}{4}$ **(coupl const.)** $\beta \in \mathbb{R}$, **param** $\propto$ **(period)**$^{-1}$.
- Symmetry: E_8 Weyl group (order $= 696729600$)
- The configuration space is the principal E_8 Weyl alcove.
- **Ground State**: $\Psi_0 = \left(\Delta_+^{(8)}\Delta_-^{(8)}\right)^{\nu}\Delta_{E_8}^{\nu}\ ,\quad E_0 = 310\beta^2\,\nu^2\ ,$

E_8 trig Hamiltonian in 8-dimensional Euclidean space with coords $x_1, x_2, \dots x_8$

$$\mathcal{H}_{E_8} = -\frac{1}{2}\Delta^{(8)} + \frac{g\beta^2}{4}\sum_{j<i=1}^{8}\left[\frac{1}{\sin^2\frac{\beta}{2}(x_i+x_j)} + \frac{1}{\sin^2\frac{\beta}{2}(x_i-x_j)}\right]$$

$$+\frac{g\beta^2}{4}\sum_{\{\nu_j\}}\frac{1}{\left[\sin^2\frac{\beta}{4}\left(x_8+\sum_{j=1}^{7}(-1)^{\nu_j}x_j\right)\right]}\ ,\ \nu_j = 0,1,\ \sum_{j=1}^{7}\nu_j \text{ is even}$$

- $g = \nu(\nu-1) > -\frac{1}{4}$ **(coupl const.)** $\beta \in \mathbb{R}$, **param** $\propto$ **(period)**$^{-1}$.
- Symmetry: E_8 Weyl group (order $= 696729600$)
- The configuration space is the principal E_8 Weyl alcove.
- **Ground State**: $\Psi_0 = (\Delta_+^{(8)}\Delta_-^{(8)})^{\nu}\Delta_{E_8}^{\nu}\ ,\qquad E_0 = 310\beta^2\nu^2\ ,$

$$\Delta_{\pm}^{(8)} = \prod_{j<i=1}^{8}\sin\frac{\beta}{2}(x_i \pm x_j)\ ,\quad \Delta_{E_8} = \prod_{\{\nu_j\}}\sin\frac{\beta}{4}(x_8+\sum_{j=1}^{7}(-1)^{\nu_j}x_j)\ .$$

- Assume factorization form $\Psi = \Psi_0 \, \Phi$

- Assume factorization form $\Psi = \Psi_0\,\Phi$
- Define gauge rotated Hamiltonian $h \equiv \Psi_0^{-1}(\mathcal{H} - E_0)\Psi_0$
  (for which Φ is eigenfunction.)

Fundamental Trigonometric Invariants (FTI)

- Assume factorization form $\Psi = \Psi_0\,\Phi$
- Define gauge rotated Hamiltonian $h \equiv \Psi_0^{-1}(\mathcal{H} - E_0)\Psi_0$ (for which Φ is eigenfunction.)
- We want to demonstrate that Φ is a polynomial in some variables (E_8 *Jack polynomial*),

Fundamental Trigonometric Invariants (FTI)

- Assume factorization form $\Psi = \Psi_0 \, \Phi$
- Define gauge rotated Hamiltonian $h \equiv \Psi_0^{-1}(\mathcal{H} - E_0)\Psi_0$ (for which Φ is eigenfunction.)
- We want to demonstrate that Φ is a polynomial in some variables (E_8 *Jack polynomial*),
- Fundamental Trigonometric (Weyl) Invariants (FTI):

$$\tau_a(x) = \sum_{\omega \in \Omega_a} e^{i\beta(w\cdot x)} \,, \quad w{=}\sum_i w_i e_i \,, x{=}\sum_i x_i e_i \in \mathbb{R}^8, \quad e_i{=}\text{std basis}$$

Ω_a: the orbit generated by fundamental weight w_a , $a{=}1,\ldots 8$

Fundamental Trigonometric Invariants (FTI)

- Assume factorization form $\Psi = \Psi_0\,\Phi$
- Define gauge rotated Hamiltonian $h \equiv \Psi_0^{-1}(\mathcal{H} - E_0)\Psi_0$ (for which Φ is eigenfunction.)
- We want to demonstrate that Φ is a polynomial in some variables (E_8 *Jack polynomial*),
- Fundamental Trigonometric (Weyl) Invariants (FTI):

$$\tau_a(x) = \sum_{\omega\in\Omega_a} e^{i\beta(w\cdot x)}\,, \quad w=\sum_i w_i e_i\,, x=\sum_i x_i e_i \in\mathbb{R}^8, \quad e_i = \text{std basis}$$

Ω_a: the orbit generated by fundamental weight w_a , $a=1,\ldots 8$

FTI and weights w_a ordered by their lengths

FTI	weight vector	w_a^2	orbit size $\lvert\Omega_a\rvert$
τ_1	$w_1 = W_8 = e_7 + e_8$	2	240
...	...	...	...
τ_8	$w_8 = W_4 = e_3 + e_4 + e_5 + e_6 + e_7 + 5e_8$	30	483840

We have shown that

(i) the similarity-transformed O-P Trigonometric Hamiltonian $h_{E_8} \propto \Psi_0^{-1}(\mathcal{H}_{E_8} - E_0)\Psi_0$, written in terms of the τ_a's is an operator in **algebraic form**:

$$h_{E_8}(\tau) = \sum_{i,j=1}^{r} A_{ij}(\tau)\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \sum_{i=1}^{r} B_i(\tau)\frac{\partial}{\partial\tau_i} \ ,$$

i.e. $A_{ij}(\tau)$, $B_i(\tau)$ are polynomials in τ_a's , $a{=}1...8$

We have shown that

(i) the similarity-transformed O-P Trigonometric Hamiltonian $h_{E_8} \propto \Psi_0^{-1}(\mathcal{H}_{E_8} - E_0)\Psi_0$, written in terms of the τ_a's is an operator in **algebraic form**:

$$h_{E_8}(\tau) = \sum_{i,j=1}^{r} A_{ij}(\tau)\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \sum_{i=1}^{r} B_i(\tau)\frac{\partial}{\partial\tau_i} \ ,$$

i.e. $A_{ij}(\tau)$, $B_i(\tau)$ are polynomials in τ_a's , $a{=}1...8$

(ii) the Hamiltonian $h_{E_8}(\tau)$ is **exactly solvable**:

$$P_n \equiv \langle \tau_1^{n_1}\tau_2^{n_2}\cdots\tau_8^{n_8} | 0 \le 2n_1{+}2n_2{+}3n_3{+}3n_4{+}4n_5{+}4n_6{+}5n_7{+}6n_8 \le n \rangle$$

are invariant subspaces of h_{E_8} $(n, n_i \in \mathbb{N})$

We have shown that

(i) the similarity-transformed O-P Trigonometric Hamiltonian $h_{E_8} \propto \Psi_0^{-1}(\mathcal{H}_{E_8} - E_0)\Psi_0$, written in terms of the τ_a's is an operator in **algebraic form**:

$$h_{E_8}(\tau) = \sum_{i,j=1}^{r} A_{ij}(\tau)\frac{\partial^2}{\partial\tau_i\partial\tau_j} + \sum_{i=1}^{r} B_i(\tau)\frac{\partial}{\partial\tau_i} ,$$

i.e. $A_{ij}(\tau)$, $B_i(\tau)$ are polynomials in τ_a's , $a{=}1\ldots 8$

(ii) the Hamiltonian $h_{E_8}(\tau)$ is **exactly solvable**:

$$P_n \equiv \langle \tau_1^{n_1}\tau_2^{n_2}\cdots\tau_8^{n_8} | 0 \le 2n_1{+}2n_2{+}3n_3{+}3n_4{+}4n_5{+}4n_6{+}5n_7{+}6n_8 \le n \rangle$$

are invariant subspaces of h_{E_8} $(n, n_i \in \mathbb{N})$

The spaces P_n form an infinite *flag*: $P_0 \subset P_1 \subset \ldots \subset P_n \subset \ldots$,

Characteristic vector: $(2, 2, 3, 3, 4, 4, 5, 6)$ $(\neq$ rational case)

$$\phi_{[1,0,0,0,0,0,0,0]} = \tau_1 + \frac{240\nu}{29\,\nu+1}\,, \qquad -\epsilon = 2 + 58\,\nu\,,$$

$$\phi_{[0,1,0,0,0,0,0,0]} = \tau_2 + \frac{126\nu}{17\,\nu+1}\tau_1 + \frac{15120\nu^2}{(17\,\nu+1)(23\,\nu+1)}\,, \qquad -\epsilon = 4 + 92\,\nu\,,$$

$$\phi_{[0,0,1,0,0,0,0,0]} = \tau_3 + \frac{84\nu\,(74\,\nu+1)}{(11\,\nu+1)(14\,\nu+1)}\tau_1 + \frac{84\nu}{11\,\nu+1}\tau_2 + \frac{6720\nu^2(74\,\nu+1)}{(14\,\nu+1)(19\,\nu+1)(11\,\nu+1)}\,, \qquad -\epsilon = 6 + 114\,\nu\,,$$

Spectrum: $-\epsilon_{\{p_1,\ldots,p_8\}} = (\mathbf{p},\mathbf{p}) + 2(\mathbf{p},\rho)\nu\,,$ (R. Sasaki et al. 2000)

$$\phi_{[1,0,0,0,0,0,0,0]} = \tau_1 + \frac{240\nu}{29\,\nu+1}\,, \qquad -\epsilon = 2 + 58\,\nu\,,$$

$$\phi_{[0,1,0,0,0,0,0,0]} = \tau_2 + \frac{126\nu}{17\,\nu+1}\tau_1 + \frac{15120\nu^2}{(17\,\nu+1)(23\,\nu+1)}\,, \qquad -\epsilon = 4 + 92\,\nu\,,$$

$$\phi_{[0,0,1,0,0,0,0,0]} = \tau_3 + \frac{84\nu\,(74\,\nu+1)}{(11\,\nu+1)(14\,\nu+1)}\tau_1 + \frac{84\nu}{11\,\nu+1}\tau_2 + \frac{6720\nu^2(74\,\nu+1)}{(14\,\nu+1)(19\,\nu+1)(11\,\nu+1)}\,, \qquad -\epsilon = 6 + 114\,\nu\,,$$

Spectrum: $-\epsilon_{\{p_1,\ldots,p_8\}} = (\mathbf{p},\mathbf{p}) + 2(\mathbf{p},\rho)\nu\,,$ (R. Sasaki et al. 2000)

$\mathbf{p} = \sum_{a=1}^{8} p_a w_a\,,\quad \rho = \sum_{a=1}^{8} w_a$ (Weyl vector)

$$\phi_{[1,0,0,0,0,0,0,0]} = \tau_1 + \frac{240\nu}{29\,\nu+1}\,, \qquad -\epsilon = 2 + 58\,\nu\,,$$

$$\phi_{[0,1,0,0,0,0,0,0]} = \tau_2 + \frac{126\nu}{17\,\nu+1}\tau_1 + \frac{15120\nu^2}{(17\,\nu+1)(23\,\nu+1)}\,, \qquad -\epsilon = 4 + 92\,\nu\,,$$

$$\phi_{[0,0,1,0,0,0,0,0]} = \tau_3 + \frac{84\nu\,(74\,\nu+1)}{(11\,\nu+1)(14\,\nu+1)}\tau_1 + \frac{84\nu}{11\,\nu+1}\tau_2 + \frac{6720\nu^2(74\,\nu+1)}{(14\,\nu+1)(19\,\nu+1)(11\,\nu+1)}\,, \qquad -\epsilon = 6 + 114\,\nu\,,$$

Spectrum: $-\epsilon_{\{p_1,\ldots,p_8\}} = (\mathbf{p},\mathbf{p}) + 2(\mathbf{p},\rho)\nu\,,$ (R. Sasaki et al. 2000)

$\mathbf{p} = \sum_{a=1}^{8} p_a w_a\,, \quad \rho = \sum_{a=1}^{8} w_a$ (Weyl vector)

Conclusions

$$\phi_{[1,0,0,0,0,0,0,0]} = \tau_1 + \frac{240\nu}{29\,\nu+1}\,, \qquad -\epsilon = 2 + 58\,\nu\,,$$

$$\phi_{[0,1,0,0,0,0,0,0]} = \tau_2 + \frac{126\nu}{17\,\nu+1}\tau_1 + \frac{15120\nu^2}{(17\,\nu+1)(23\,\nu+1)}\,, \qquad -\epsilon = 4 + 92\,\nu\,,$$

$$\phi_{[0,0,1,0,0,0,0,0]} = \tau_3 + \frac{84\nu\,(74\,\nu+1)}{(11\,\nu+1)(14\,\nu+1)}\tau_1 + \frac{84\nu}{11\,\nu+1}\tau_2 + \frac{6720\nu^2(74\,\nu+1)}{(14\,\nu+1)(19\,\nu+1)(11\,\nu+1)}\,, \qquad -\epsilon = 6 + 114\,\nu\,,$$

Spectrum: $-\epsilon_{\{p_1,\ldots,p_8\}} = (\mathbf{p},\mathbf{p}) + 2(\mathbf{p},\rho)\nu\,,$ (R. Sasaki et al. 2000)
$\mathbf{p} = \sum_{a=1}^{8} p_a w_a\,, \quad \rho = \sum_{a=1}^{8} w_a$ (Weyl vector)

Conclusions

- The use of FTI enabled us to find an algebraic form of the Hamiltonian associated to E_8, which did not seem feasible at all, in the past

$$\phi_{[1,0,0,0,0,0,0,0]} = \tau_1 + \frac{240\nu}{29\,\nu+1}\,, \qquad -\epsilon = 2 + 58\,\nu\,,$$

$$\phi_{[0,1,0,0,0,0,0,0]} = \tau_2 + \frac{126\nu}{17\,\nu+1}\tau_1 + \frac{15120\nu^2}{(17\,\nu+1)(23\,\nu+1)}\,, \qquad -\epsilon = 4 + 92\,\nu\,,$$

$$\phi_{[0,0,1,0,0,0,0,0]} = \tau_3 + \frac{84\nu\,(74\,\nu+1)}{(11\,\nu+1)(14\,\nu+1)}\tau_1 + \frac{84\nu}{11\,\nu+1}\tau_2 + \frac{6720\nu^2(74\,\nu+1)}{(14\,\nu+1)(19\,\nu+1)(11\,\nu+1)}\,, \qquad -\epsilon = 6 + 114\,\nu\,,$$

Spectrum: $-\epsilon_{\{p_1,\ldots,p_8\}} = (\mathbf{p},\mathbf{p}) + 2(\mathbf{p},\rho)\nu\,,$ (R. Sasaki et al. 2000)

$\mathbf{p} = \sum_{a=1}^{8} p_a w_a\,,\quad \rho = \sum_{a=1}^{8} w_a$ (Weyl vector)

Conclusions

- The use of FTI enabled us to find an algebraic form of the Hamiltonian associated to E_8, which did not seem feasible at all, in the past
- The Olshanetsky-Perelomov Hamiltonian, in FTI algebraic form, preserves an infinite flag of polynomial spaces, with a characteristic vector $\vec{\alpha} = (2,2,3,3,4,4,5,6)$, i.e it is exactly solvable (block diagonal form).

$$\phi_{[1,0,0,0,0,0,0,0]} = \tau_1 + \frac{240\nu}{29\,\nu+1}\,, \qquad -\epsilon = 2 + 58\,\nu\,,$$

$$\phi_{[0,1,0,0,0,0,0,0]} = \tau_2 + \frac{126\nu}{17\,\nu+1}\tau_1 + \frac{15120\nu^2}{(17\,\nu+1)(23\,\nu+1)}\,, \qquad -\epsilon = 4 + 92\,\nu\,,$$

$$\phi_{[0,0,1,0,0,0,0,0]} = \tau_3 + \frac{84\nu\,(74\,\nu+1)}{(11\,\nu+1)(14\,\nu+1)}\tau_1 + \frac{84\nu}{11\,\nu+1}\tau_2 + \frac{6720\nu^2(74\,\nu+1)}{(14\,\nu+1)(19\,\nu+1)(11\,\nu+1)}\,, \qquad -\epsilon = 6 + 114\,\nu\,,$$

Spectrum: $-\epsilon_{\{p_1,\ldots,p_8\}} = (\mathbf{p},\mathbf{p}) + 2(\mathbf{p},\rho)\nu\,,$ (R. Sasaki et al. 2000)

$\mathbf{p} = \sum_{a=1}^{8} p_a w_a\,,\quad \rho = \sum_{a=1}^{8} w_a$ (Weyl vector)

Conclusions

- The use of FTI enabled us to find an algebraic form of the Hamiltonian associated to E_8, which did not seem feasible at all, in the past
- The Olshanetsky-Perelomov Hamiltonian, in FTI algebraic form, preserves an infinite flag of polynomial spaces, with a characteristic vector $\vec{\alpha} = (2,2,3,3,4,4,5,6)$, i.e it is exactly solvable (block diagonal form).

Thank you

Happy Birthday to Francesco Calogero!