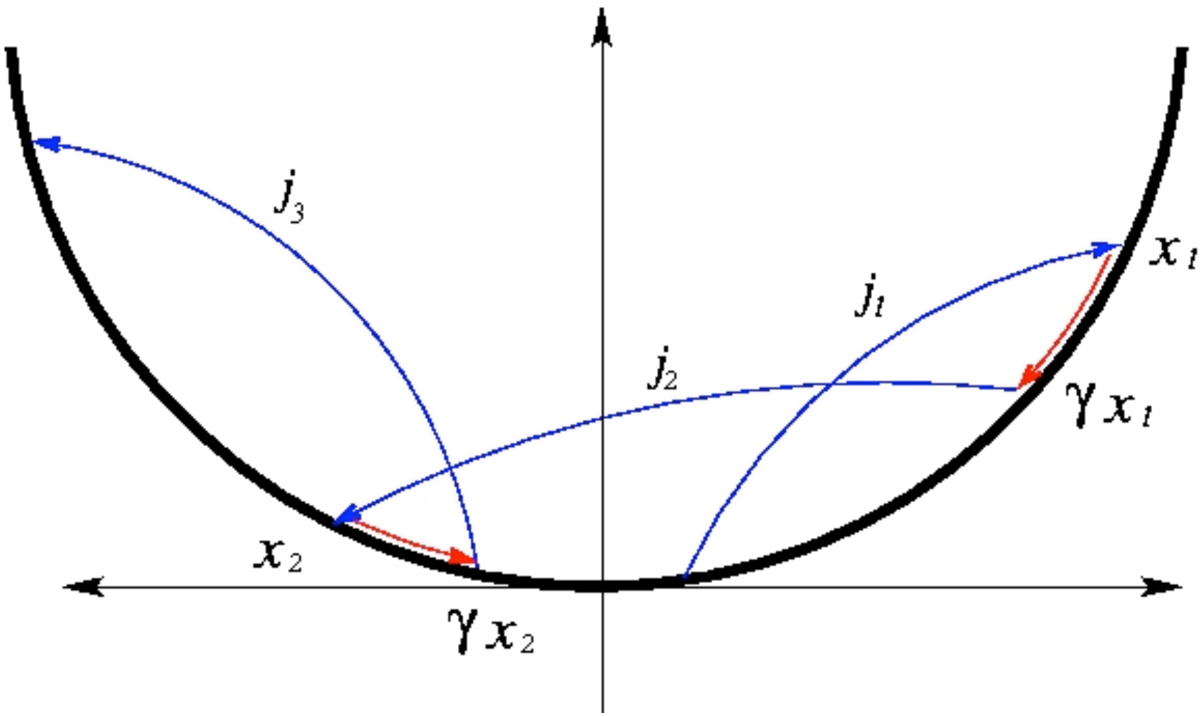
First passage time distribution for a discrete version of the Ornstein-Uhlenbeck process

Hernán Larralde

ICF-UNAM

The discrete O-U process may be thought of as a random walk that jumps and then ``slides" down a harmonic potential during a fixed time interval... Then it jumps again, and slides again... and so on.



$$x_{n+1} = j_{n+1} + \gamma x_n \qquad\qquad x_{n=0} = x_o$$

where the steps j are i.i.d. variables with zero mean

and common probability distribution: $\phi(j)$

(γ is a constant)...

The probability distribution function of x_n satisfies the recursion relation:

$$P_{n+1}(x) = \int_{-\infty}^{\infty} \phi(x - \gamma y) P_n(y) dy.$$

Assuming the jump distribution $\phi(j)$ to be sharply peaked at the origin

with second moment $\sigma^2 \sim 2D\tau$ then, taking $\gamma \sim 1 - k\tau$

the continuous O-U process is recovered

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2}{\partial x^2}P(x,t) + k\frac{\partial}{\partial x}xP(x,t) \qquad t = n\tau$$

However, the recurrence relation can be dealt with directly.

A Fourier transform casts the recursion relation into:

$$\hat{P}_{n+1}(\theta) = \hat{\phi}(\theta)\hat{P}_n(\gamma\theta) \qquad \hat{P}_0(\theta) = e^{i\theta x_o}$$

So $$\hat{P}_n(\theta) = e^{i\gamma^n \theta x_o}\prod_{m=0}^{n-1}\hat{\phi}(\gamma^m\theta) \qquad n = 1, 2, 3...$$

First passage probability...

We now consider the probability of first entrance to the negative real axis at step n, starting from the origin.

The corresponding result for the normal random walk $(\gamma = 1)$ is

$$\tau(z) = 1 - \sqrt{1 - z},$$

independent of the step distribution!!!!
(as long as this distribution is symmetric and continuous) Sparre-Anderson theorem

The trivial limit $\gamma = 0$ is universal too! $\tau(z) = \frac{z}{2 - z}$

Is it always universal? NO

...what is it then?

Denote by P_n(x) the probability density of finding the particle at position x at step n when no entrance to the negative axis has occurred.

Denote Q_n(x) as the probability density for the position x of the first entrance at step n.

These distributions satisfy the following equations:

$$P_{n+1}(x) = \begin{cases} \int_0^{\infty} \phi(x - \gamma y) P_n(y) dy & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$Q_{n+1}(x) = \begin{cases} \int_0^{\infty} \phi(x - \gamma y) P_n(y) dy & \text{if } 0 > x \\ 0 & \text{if } x \geq 0 \end{cases}$$

$$P_{n+1}(x)+Q_{n+1}(x)=\int_{-\infty}^{\infty}\phi(x-\gamma y)P_n(y)dy$$

Defining the generating function: $$P(x,z)\equiv\sum_{z=0}^{\infty}z^nP_n(x)$$

.....and Fourier transforming yields:

$$\hat{P}^{+}(\theta,z)+\hat{Q}^{-}(\theta,z)=1+z\hat{\phi}(\theta)\hat{P}^{+}(\gamma\theta,z).$$

...a “deformed” Wiener Hopf equation... which I cannot factorize for arbitrary γ

If I could.... $$\tau(z)=\hat{Q}^{-}(\theta=0,z)$$

But there is one solvable case!!!

$$\phi(j) = \frac{1}{2\lambda} e^{-\lambda|x|} \qquad \lambda = 1$$

The generating function of the recurrence relation is:

$$P(x,z) = \frac{z}{2} e^{-x} + \frac{z}{2} \int_0^\infty e^{-|x-\gamma y|} P(y,z) dy, \qquad (x \geq 0)$$

Propose:

$$P(x,z) = \sum_{m=0}^{\infty} A_m(z) e^{-\frac{x}{\gamma^m}}$$

Then the coefficients satisfy:

$$A_m(z) = -\frac{z}{\gamma} \frac{\gamma^{2m}}{1-\gamma^{2m}} A_{m-1}(z) \qquad A_0(z) = \frac{z}{2} + \frac{z}{2} \sum_{m=0}^{\infty} \frac{A_m(z)\gamma^m}{1-\gamma^{m+1}}$$

This procedure leads to:

$$\tau(z) = \frac{z}{2}\,\frac{1 - z\left[\frac{\gamma}{1-\gamma^2} + \sum_{m=1}^{\infty}(-z)^m \frac{\gamma^{(m+1)^2}}{\prod_{\nu=1}^{(m+1)}(1-\gamma^{2\nu})}\right]}{1 - \frac{z}{2}\left[\frac{1}{1-\gamma} + \sum_{m=1}^{\infty}(-z)^m \frac{\gamma^{m^2+m}}{(1-\gamma^{m+1})\prod_{\nu=1}^{m}(1-\gamma^{2\nu})}\right]}$$

Oh my god!!!!!!!!!!!

....but, remarkably, this can be written as:

$$\tau(z) = \frac{z(\gamma z, \gamma^2)_\infty}{(\gamma z, \gamma^2)_\infty + (z, \gamma^2)_\infty}$$

$$(z, q)_\infty = \prod_{\nu=0}^{\infty}(1 - zq^\nu)$$

From this expression, the moments of the first passage distribution can be calculated directly... For example:

$$\langle n \rangle_\gamma = \lim_{z \to 1} \frac{d\tau(z)}{dz} = 1 + \frac{(\gamma^2;\gamma^2)_\infty}{(\gamma;\gamma^2)_\infty}$$

Also, using $\lim_{q \to 1^-} \frac{(q^\lambda x, q)_\infty}{(q^\mu x, q)_\infty} = (1-x)^{\mu-\lambda}$ (ask Natig!)

We recover:

$$\lim_{\gamma \to 1^-} \tau(z) = \frac{z}{1+\sqrt{1-z}} = 1 - \sqrt{1-z}$$

... asymptotics...

For example:

...Singularities of $\tau(z)$ Consider the denominator:

$$D(z) = \prod_{\nu=0}^{\infty}(1 - z\gamma^{2\nu+1}) + \prod_{\nu=0}^{\infty}(1 - z\gamma^{2\nu}) > 0 \qquad \text{for} \qquad z < 1$$

$D(\gamma^{-1}) < 0$ so $\tau(n)$ decays exponentially at large n...

The first passage probability of the discrete O-U does not share the universality properties of Sparre Andersen's theorem

Thus, an explicit calculation is required for each choice of jump distributions... and the general solution does not seem to be simple.

An exact closed form expression for the generating function of the first passage probability can be obtained if the jump distribution is the double exponential.

Other first passage probabilities for this choice of jump distribution can be carried out in essentially the same way... For example:

- -The first passage probability from the origin to the region beyond a point x (the counterpart of the Kramers problem)

- -The “arcsine law” for this process, etc.

However, a more satisfying achievement would be to find the general solution to the Wiener-Hopf equation for arbitrary jump distributions... Homework anybody?