The initial value problem for integrable matrix partial differential equations in 1+1 dimensions

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Outline

- Introduction to integrable field equations in 1+1 dimensions
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- Generalisation to equations with non commuting coefficients
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- Solvability of the initial value problem
-
- Example: two wave non local interaction model

Background 1: OLD TIMES

Lax pair : $$\Psi_x = V\Psi \quad , \quad \Psi_t = A\Psi$$

$$V = V(x, t, k) \qquad A = A(x, t, k)$$

$$V\Psi dx + A\Psi dt = \text{ exact one-form}$$

Compatibility : $$[V\,,\,A] + V_t - A_x = 0$$

V and A are 2×2 matrix valued differential polynomials of the field variable $u(x, t)$ and algebraic polynomials of the spectral variable k

Korteweg-de Vries equation $\quad u_t = u_{xxx} - 6uu_x$

Nonlinear Schroedinger equation $\quad u_t = i(u_{xx} \pm 2|u|^2 u)$

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Background 2: OLD TIMES

Spectral transform as reflection coefficient $R(k,t)$

Direct spectral problem : $u(x,t) \to R(k,t)$

Inverse spectral problem : $R(k,t) \to u(x,t)$

Solving the initial value problem $u(x,0) \to u(x,t)$ by the spectral method

$$u(x,0) \to R(k,0) \to R(k,t) \to u(x,t)$$

$$R_t = -i\omega(k)R$$

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Background 3: OLD TIMES

1) Hierarchy of infinitely many flows

Example: KdV hierarchy $u_{t_n} = L^n u_x \;,\;\; n = 0\,,\,1\,,\,2\,,\cdots\,,$

$$Lf(x) = f_{xx}(x) - 4uf(x) + 2u_x \int_x^{+\infty} dy f(y)$$

$$u_t = \sum_{n=0} c_n\, L^n u_x$$

1a) $L^n u_x$ is a differential polynomial of u

1b) commutativity of flows $u_{t_n t_m} = u_{t_m t_n}$ for any $n\,,\; m$

2) Infinitely many independent conservation laws

Background 4: OLD TIMES

MATRIX KORTEWEG – DEVRIES EQUATION

$$U_t = U_{xxx} - 3\{U\,,\,U_x\}$$

$$U_t = \sum_{n=0} c_n L^n U_x$$

MATRIX NONLINEAR SCROEDINGER EQUATION

$$U_t = i(U_{xx} \pm 2\,U\,U^{\dagger}\,U)$$

NON COMMUTING COEFFICIENTS

$$V(k) = ik\sigma + Q(x,t)$$

$$\sigma = \text{block–diag}(\sigma_1 \mathbf{1}_1, \sigma_2 \mathbf{1}_2, \cdots, \sigma_M \mathbf{1}_M)\ ,\ \mathbf{1}_j = N_j x N_j \ \text{ identity}$$

$$\sigma_j \neq \sigma_m\ ,\ \text{if } j \neq m$$

$$Q_t = \sum_n L^n [C^{(n)} Q]$$

$$C^{(n)} = \text{block–diag}(C_1^{(n)}, C_2^{(n)}, \cdots, C_M^{(n)})$$

consequences:

1. noncommuting flows
2. nonlocal (integro–differential) evolution equations
3. solitons have a time–dependent velocity

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EXAMPLE : THREE WAVE RESONANT INTERACTION EQUATION

$$u_t^{(n)} + V_n u_x^{(n)} + \eta_n u^{(n+1)*} u^{(n+2)*} = 0 \ , \ \ n = 1, 2, 3 \ , \ \mathrm{mod}\ 3$$

BOOMERONS IN NONLINEAR OPTICS

FIRST EXPERIMENTAL OBSERVATION OF SIMULTONS
Phys. Rev. Lett. 104, 113902 (2010)
F. Baronio, M. Conforti, C. De Angelis, A. Degasperis, M. Andreana, V. Couderc, and A. Barthelemy

NON LOCAL WAVE INTERACTION

$$f^{(1)}_t - f^{(1)}_x = -g f^{(2)}\,, \quad f^{(2)}_t + f^{(2)}_x = g^* f^{(1)}\,,$$

$$g(x,t) = g_0(t) + \int_{x_0}^{x} dy\; f^{(2)*}(y,t)\; f^{(1)}(y,t)$$

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \;, \quad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & f^{(1)*} & f^{(2)*} \\ f^{(1)} & 0 & 0 \\ f^{(2)} & 0 & 0 \end{pmatrix}$$

$$\Psi_t = (2ikC - \sigma W + \sigma[C, Q(x,t)])\,\Psi \;, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,,$$

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -g \\ 0 & g^* & 0 \end{pmatrix} \,.$$

SPECTRAL METHOD : DIRECT PROBLEM

$$\begin{cases} \Psi_L \to \exp(ikx\sigma) \ , \quad x \to -\infty \\ \Psi_L \to \exp(ikx\sigma) S_L(k,t) \ , \quad x \to +\infty \ . \end{cases}$$

$$\begin{cases} \Psi_R \to \exp(ikx\sigma) \ , \quad x \to +\infty \\ \Psi_R \to \exp(ikx\sigma) S_R(k,t) \ , \quad x \to -\infty \ . \end{cases}$$

$$S_L = (1 + R_L) T_L^{-1} \ , \quad S_R = (1 + R_R) T_R^{-1} \ ,$$

$$R_L = \begin{pmatrix} 0 & R_L^{(1)*} & R_L^{(2)*} \\ R_L^{(1)} & 0 & 0 \\ R_L^{(2)} & 0 & 0 \end{pmatrix} \ , \quad R_R = \begin{pmatrix} 0 & R_R^{(1)*} & R_R^{(2)*} \\ R_R^{(1)} & 0 & 0 \\ R_R^{(2)} & 0 & 0 \end{pmatrix}$$

$$T_L = \begin{pmatrix} T_{L11} & 0 & 0 \\ 0 & T_{L22} & T_{L23} \\ 0 & T_{L32} & T_{L33} \end{pmatrix} \ , \quad T_R = \begin{pmatrix} T_{R11} & 0 & 0 \\ 0 & T_{R22} & T_{R23} \\ 0 & T_{R32} & T_{R33} \end{pmatrix}$$

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SPECTRAL METHOD : INVERSE PROBLEM

$$M_L(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{ikx\sigma} R_L(k) \;\;,\;\; M_R(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{ikx\sigma} R_R(k)$$

$$K_L(x,y) + M_L(x+y) + \int_x^{+\infty} dz K_L(x,z) M_L(z+y) = 0 \;\;,\;\; x < y \;\;,$$

$$K_R(x,y) + M_R(x+y) + \int_{-\infty}^{x} dz K_R(x,z) M_R(z+y) = 0 \;\;,\;\; x > y \;\;.$$

$$Q(x) = -\sigma[\sigma\,,\, K_L(x,x)] = \sigma[\sigma\,,\, K_R(x,x)]$$

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INITIAL BOUNDARY VALUE PROBLEM

$f^{(1)}(x,0)$ and $f^{(2)}(x,0)$ are given, and also $g(x_0,t) = g_0(t)$ is given

$$\Psi_t = (2ikC - \sigma W + \sigma[C, Q(x,t)])\,\Psi$$

$$R_{Lt} = [A_+ \, , \, R_L] \quad , \quad R_{Rt} = [A_- \, , \, R_R] \quad ,$$

$$A_{\pm} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2ik & -g_{\pm} \\ 0 & g_{\pm}^* & 2ik \end{pmatrix}$$

$$g_{\pm}(t) = g_0(t) + \int_{x_0}^{\pm\infty} dx f^{(2)*}(x,t) f^{(1)}(x,t)$$

$$A_- = A_+ + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (f^{(2)}\, , \, f^{(1)}) \\ 0 & -(f^{(1)}\, , \, f^{(2)}) & 0 \end{pmatrix}$$

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SOLVABLE INITIAL BOUNDARY VALUE PROBLEM

$$x_0 = +\infty \quad , \quad g_+(t) = g_0(t)$$

$$\begin{pmatrix} R_L^{(2)} \\ R_L^{(1)} \end{pmatrix}_t = \begin{pmatrix} 2ik & g_0^*(t) \\ -g_0(t) & -2ik \end{pmatrix} \begin{pmatrix} R_L^{(2)} \\ R_L^{(1)} \end{pmatrix}$$

$$x_0 = -\infty \quad , \quad g_-(t) = g_0(t)$$

$$\begin{pmatrix} R_R^{(2)} \\ R_R^{(1)} \end{pmatrix}_t = \begin{pmatrix} 2ik & g_0^*(t) \\ -g_0(t) & -2ik \end{pmatrix} \begin{pmatrix} R_R^{(2)} \\ R_R^{(1)} \end{pmatrix}$$

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SOLVABLE SPECTRAL DATA NONLINEAR EVOLUTION EQUATION

$$x_0 = +\infty \ , \ \ g_-(t) = g_0(t) - (f^{(2)}\, , \, f^{(1)})$$

$$\begin{pmatrix} R_R^{(2)} \\ R_R^{(1)} \end{pmatrix}_t = \begin{pmatrix} 2ik & g_-^*(t) \\ -g_-(t) & -2ik \end{pmatrix} \begin{pmatrix} R_R^{(2)} \\ R_R^{(1)} \end{pmatrix}$$

$$(f^{(2)}\, , \, f^{(1)}) = \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \, \mu(k_1, k_2, t) R_R^{(2)*}(k_2, t) \, R_R^{(1)}(k_1, t)$$

$$R_R^{(1)}(k,0)\, , \ R_R^{(2)}(k,0) \to R_R^{(1)}(k,t)\, , \ R_R^{(2)}(k,t)$$

$$R_R(k,0) \to Q(x,0) \to R_L(k,0) \to R_L(k,t) \to Q(x,t) \to R_R(k,t)$$

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$$R_R^{(1)}(k, 0) \, , \ R_R^{(2)}(k, 0) \rightarrow R_R^{(1)}(k, t) \ , \ R_R^{(2)}(k, t)$$

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$$R_R(k,0) \to Q(x,0) \to R_L(k,0) \to R_L(k,t) \to Q(x,t) \to R_R(k,t)$$

MORE EVOLUTION

..........WITH MY VERY BEST WISHES TO FRANCESCO AND.......................

CENTO DI QUESTI GIORNI !!!!