An invertible transformation and some of its applications

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Abstract

An explicitly invertible transformation is reported, and several of its applications. This transformation is elementary and therefore all the results obtained via it might be considered *trivial*; yet the findings described in this report are generally far from appearing trivial until the way they are obtained is revealed. Various contexts can be considered: algebraic and Diophantine equations, nonlinear Sturm-Liouville problems, dynamical systems (with continuous and with discrete time), nonlinear partial differential equations, analytical geometry, functional equations, etc. etc. While this transformation, in one or another context, is certainly known to many, it does not seem to be as universally known as it deserves to be, for instance it is not routinely taught in basic University courses (to the best of our knowledge). Some generalizations of this transformation are also reported.

All these results have been obtained in collaboration with **Mario Bruschi, François Leyvraz** and **Matteo Sommacal**.

They are reported in the following 2 papers:

M. Bruschi, F. Calogero, F. Leyvraz and M. Sommacal, "An invertible transformation and some of its applications", J. Nonlinear Math. Phys. (in press); "Generalization of an invertible transformation and examples of its applications" (in preparation).

The *explicitly invertible* transformation

It consists of a change of variables, involving 2 arbitrary functions $F_1(w), F_2(w)$, from 2 quantities u_1, u_2, to 2 quantities x_1, x_2 and viceversa. It reads as follows:

$$x_1 = u_1 + F_1(u_2), \; x_2 = u_2 + F_2(x_1) = u_2 + F_2\big(u_1 + F_1(u_2)\big),$$
$$u_1 = x_1 - F_1(u_2) = x_1 - F_1\big(x_2 - F_2(x_1)\big), \; u_2 = x_2 - F_2(x_1).$$

The most remarkable aspect of this transformation is its explicitly invertible character: note that both the direct respectively the inverse changes of variables involve *only* (albeit also in a nested manner) the 2 arbitrary functions $F_1(w), F_2(w)$, and *not* their inverses. This in particular entails that, if the 2 functions $F_1(w), F_2(w)$ are *one-valued* (as we hereafter assume), both the direct and inverse changes of variables are *one-valued*; if the 2 functions $F_1(w), F_2(w)$ are *entire*, this property is inherited by both the direct and inverse changes of variables; if the 2 functions $F_1(w), F_2(w)$ are *polynomials* (of arbitrary degree), both the expressions of x_1, x_2 in terms of u_1, u_2, and the expressions of u_1, u_2, in terms of x_1, x_2, are as well *polynomial*.

Remark: the above transformation can be obtained as a composition of two *triangular* “seed” transformations:

$$y_1 = u_1 + F_1(u_2), \;\; y_2 = u_2\,,$$

$$x_1 = y_1\,, \;\; x_2 = y_2 + F_2(y_1),$$

clearly entailing

$$x_1 = u_1 + F_1(u_2),$$

$$x_2 = u_2 + F_2(x_1) = u_2 + F_2\big(u_1 + F_1(u_2)\big).$$

It can be moreover easily checked that this is an *area-preserving* transformation:

$$\begin{vmatrix} \dfrac{\partial x_1}{\partial u_1} & \dfrac{\partial x_1}{\partial u_2} \\ \dfrac{\partial x_2}{\partial u_1} & \dfrac{\partial x_{21}}{\partial u_{21}} \end{vmatrix} = 1\,.$$

Example: for instance for

$$F_1(w) = c_1\, w^2\,,\;\; F_2(w) = c_2\, w^2\,,$$

the direct and inverse transformations read as follows:

$$x_1 = u_1 + c_1\, u_2^{\;2}\,,\;\; x_2 = u_2 + c_2\left(u_1 + c_1\, u_2^{\;2}\right)^2,$$

$$u_1 = x_1 - c_1\left(x_2 - c_2\, x_1^{\;2}\right)^2,\;\; u_2 = x_2 - c_2\, x_1^{\;2}\,.$$

Generalizations

A multinested approach: 2 variables, more than 2 arbitrary functions

3 arbitrary functions

$$x_1 = u_1 + F_1(u_2) + F_3(u_2 + F_2(u_1 + F_1(u_2))),$$
$$x_2 = u_2 + F_2(u_1 + F_1(u_2));$$

$$u_1 = x_1 - F_3(x_2) - F_1(x_2 - F_2(x_1 - F_3(x_2))),$$
$$u_2 = x_2 - F_2(x_1 - F_3(x_2)).$$

4 arbitrary functions

$$x_1 = u_1 + F_1(u_2) + F_3\left(u_2 + F_2\left(u_1 + F_1(u_2)\right)\right),$$

$$x_2 = u_2 + F_2\left(u_1 + F_1(u_2)\right)$$

$$+ F_4\left(u_1 + F_1(u_2)\right) + F_3\left(u_2 + F_2\left(u_1 + F_1(u_2)\right)\right),$$

$$u_1 = x_1 - F_3\left(x_2 - F_4(x_1)\right) - F_1\left(x_2 - F_4(x_1)\right)$$

$$- F_2\left(x_2 - F_3\left(x_2 - F_4(x_1)\right)\right),$$

$$u_2 = x_2 - F_4(x_1) - F_2\left(x_1 - F_3\left(x_2 - F_4(x_1)\right)\right).$$

N arbitrary functions

Can be written recursively, as extrapolation of those written above.

More variables

A direct approach

***N*=3**:

$$x_1 = u_1 + F_1(u_2, u_3),\ \ x_2 = u_2 + F_2(x_1, u_3),\ \ x_3 = u_3 + F_3(x_1, x_2);$$

$$u_3 = x_3 - F_3(x_1, x_2),\ \ u_2 = x_2 - F_2(x_1, u_3),\ \ u_1 = x_1 - F_1(u_2, u_3).$$

***N*=4:**

$$x_1 = u_1 + F_1(u_2, u_3, u_4),\ \ x_2 = u_2 + F_2(x_1, u_3, u_4),$$

$$x_3 = u_3 + F_3(x_1, x_2, u_4),\ \ x_4 = u_4 + F_4(x_1, x_2, x_3);$$

$$u_4 = x_4 - F_4(x_1, x_2, x_3),\ \ u_3 = x_3 - F_3(x_1, x_2, u_4),$$

$$u_2 = x_2 - F_2(x_1, u_3, u_4),\ \ u_1 = x_1 - F_1(u_2, u_3, u_4).$$

And the formulas for arbitrary *N* are then obvious.

Matrices

This generalization to more variables (say, to N^2 variables) is quite straightforward, amounting to a systematic replacement of scalars with $N \times N$ matrices. Of course while doing so appropriate account must be taken of the *noncommutativity* of matrices.

A more general generalization

(reported here for simplicity for only 2 variables)

To arrive at our “more general” generalization we take as point of departure two assumedly known invertible transformations, which we write in operatorial form as follows:

$$z = T_n \cdot y\,,\;\; y = T_n^{-1} \cdot z\,, \quad n = 1,2,\ldots$$

And let us assume that the direct and inverse versions of each of these transformations depend on an arbitrary number of parameters f_{nk}, which themselves may be functions of another variable *u*:

$$T_n \equiv T_n\left(f_{nk}\right),\;\; f_{nk} \equiv f_{nk}(u).$$

For instance a simple example of invertible transformation ("Möbius"), from *y* to *z* and viceversa, reads as follows:

$$z=\frac{y\,f_1(u)+f_2(u)}{y\,f_3(u)+f_4(u)},\;\; y=\frac{z\,f_4(u)-f_2(u)}{z\,f_3(u)-f_1(u)},$$

where the 4 a priori arbitrary functions $f_{nk}(u)$ are only restricted by the condition that the combination $D(u)=f_1(u)\,f_4(u)-f_2(u)\,f_3(u)$ not vanish *identically*.

Let us now consider the transformation---from 2 quantities u_1, u_2 to 2 quantities x_1, x_2 ---reading as follows:

$$x_1 = T_1\left(f_{1k}(u_2)\right) \cdot u_1 ,$$

$$x_2 = T_2\left(f_{2k}(x_1)\right) \cdot u_2 = T_2\left(f_{2k}\left(T_1\left(f_{1k}(u_2)\right) \cdot u_1\right)\right) \cdot u_2 ,$$

which---under the above assumptions---can clearly be explicitly inverted, to read

$$u_2 = T_2^{-1}\left(f_{2k}(x_1)\right) \cdot x_2 ,$$

$$u_1 = T_1^{-1}\left(f_{1k}(u_2)\right) \cdot x_1 = T_1^{-1}\left(f_{1k}\left(T_2^{-1}\left(f_{2k}(x_1)\right) \cdot x_2\right)\right) \cdot x_1 .$$

Note that both the direct transformation and the inverse transformation involve the functions $f_{nk}(u)$ but *not* their inverses.

For instance if the 2 transformations T_n are both of Möbius type, then the direct transformation reads

$$x_1 = \frac{u_1\ f_{11}(u_2) + f_{12}(u_2)}{u_1\ f_{13}(u_2) + f_{14}(u_2)}, \quad x_2 = \frac{u_2\ f_{21}(x_1) + f_{22}(x_1)}{u_2\ f_{23}(x_1) + f_{24}(x_1)},$$

and the inverse transformation reads

$$u_1 = -\frac{x_1\ f_{14}(u_2) - f_{12}(u_2)}{x_1\ f_{13}(u_2) - f_{11}(u_2)}, \quad u_2 = -\frac{x_2\ f_{24}(x_1) - f_{22}(x_1)}{x_2\ f_{23}(x_1) - f_{21}(x_1)}.$$

Note that these *explicit* transformations involve 8 *a priori arbitrary* functions $f_{nk}(w)$.

Remark: above we assumed the functions $f_{nk}(w)$ to depend on a *single* argument, and the generalized transformation to relate 2 quantities u_1, u_2 to 2 quantities x_1, x_2 and viceversa. The extension of this treatment to *explicitly invertible* transformations from *N* quantities u_n to *N* quantities x_n, and viceversa, with *N>2*, is rather obvious; they will involve functions of *N-1* arguments.

Remark: the various generalizations described above can of course be combined.

Applications: some representative examples

Algebraic and Diophantine equations

Assume that the 2 quantities u_1, u_2 are the solutions of the following 2 quite trivial algebraic equations:

$$u_1 = u_2 \,, \; u_1 u_2 + \alpha u_1 + \beta u_2 + \gamma = 0 \,,$$

obviously implying

$$u_1 = u_2 = u_\pm = \frac{1}{2}\left(-\alpha - \beta \pm \sqrt{(\alpha + \beta)^2 - 4\gamma}\right).$$

Then, by using the simplest of our invertible transformations, we conclude that the following system of 2 equations in the 2 unknowns x_1, x_2 ,

$$x_1 - F_1\left(x_2 - F_2\left(x_1\right)\right) = x_2 - F_2\left(x_1\right),$$

$$\left[x_1 - F_1\left(x_2 - F_2\left(x_1\right)\right)\right]\left[x_2 - F_2\left(x_1\right)\right]$$
$$+\alpha\left[x_1 - F_1\left(x_2 - F_2\left(x_1\right)\right)\right] + \beta\left[x_2 - F_2\left(x_1\right)\right] + \gamma = 0,$$

where $F_1(w), F_2(w)$ are 2 *arbitrary* functions, has the two solutions

$$x_1 = u_\pm + F_1\left(u_\pm\right),\ \ x_2 = u_\pm + F_2\left(u_\pm + F_1\left(u_\pm\right)\right).$$

Remark: if $\alpha + \beta = j$, $(\alpha + \beta)^2 - 4\gamma = k^2$ with *j*, *k* two *arbitrary* integers, and the 2 functions $F_1(w), F_2(w)$ are two *arbitrary polynomials* with *integer* coefficients, the above system of 2 equations in the 2 unknowns x_1, x_2 is *Diophantine* !

Nonlinear Sturm-Liouville problems

An example of highly nonlinear Sturm-Liouville problem reads as follows:

$$x_1' = x_2 + x_1\left[\left(4\alpha\lambda-\beta\right)x_1+4\alpha z x_2\right]-4\alpha\left(\beta z {x_1}^3+2\alpha\lambda x_1 {x_2}^2+\alpha z {x_2}^3\right)$$
$$+4\alpha^2\left[\ x_2\left(4\beta\lambda {x_1}^3+3\beta z {x_1}^2 x_2+\alpha\lambda {x_2}^3\right)-\beta {x_1}^2\left(2\beta\lambda {x_1}^3+3\beta z {x_1}^2 x_2+\alpha\lambda {x_2}^3\right)\right.$$
$$\left.+\beta^2 {x_1}^4\left(\beta z {x_1}^2+6\alpha\lambda {x_2}^2\right)-4\alpha\beta^3\lambda {x_1}^6 x_2+\alpha\beta^4\lambda {x_1}^8\ \right],$$

$$x_2' = 2\left(\lambda x_1 + z x_2 - \beta z {x_1}^2 - \alpha\lambda {x_2}^2 + 2\alpha\beta\lambda {x_1}^2 x_2 - \alpha\beta^2\lambda {x_1}^4\right)+2\beta x_1\, x_1'.$$

Here and below z is the (independent) variable, $x_1 \equiv x_1(z), x_2 \equiv x_2(z)$ are two functions of this variable, appended primes indicate differentiation with respect to this variable, α,β are two *arbitrary* constants and λ is the *eigenvalue* of the nonlinear Sturm-Liouville problem characterized by this system of two first-order nonlinear ODEs and the requirement that the eigenfunctions $x_1(z), x_2(z)$ be *polynomials* in the variable z. The solution of this problem is that the eigenvalues λ are the *nonnegative integers*, $\lambda=\ell,\ \ell=0,1,2,...,$ and the corresponding eigenfunctions are

$$x_1(z) = H_\ell(z) + 4\alpha\left[\ell H_{\ell-1}(z)\right]^2,$$

$$x_2(z) = 2\ell H_{\ell-1}(z) + \beta\left\{H_\ell(z) + 4\alpha\left[\ell H_{\ell-1}(z)\right]^2\right\}.$$

Here and below $H_\ell(z)$ is the Hermite polynomial of degree ℓ in the variable z.

These findings are obtained from the standard linear Sturm-Liouville problem characterizing Hermite polynomials, reading

$$u'' - 2zu' - 2\lambda u = 0,\ u \equiv u(z),$$

And---via the requirement that $u(z)$ be a polynomial in z entailing $\lambda = \ell,\ \ell = 0,1,2,\ldots;\ u(z) = H_\ell(z)$---by setting $u(z) = u_1(z),\ u'(z) = u_2(z)$ and by then relating $u_1(z), u_2(z)$ to $x_1(z), x_2(z)$ via our standard invertible transformation.

Dynamical systems

Let us recall that a dynamical system consists of a finite but otherwise *a priori arbitrary* number *N* of (first-order) Ordinary Differential Equations (ODEs), say

$$\dot{x}_n = f_n(\underline{x}),\ n=1,2,...,N\ ;\ \dot{\underline{x}} = \underline{f}(\underline{x}),$$

where the *N* (real) "dependent variables" $x_n \equiv x_n(t)$ are functions of the (real) "independent variable" t ("time"), superimposed dots indicate time-differentiations, $\underline{x}$ is the *N*-vector of components $x_n,\ \underline{x} \equiv (x_1,...,x_N)$, and the *N* functions $f_n(\underline{x})$ are (*a priori arbitrarily*) assigned.

Here for simplicity we restrict attention to systems with *N=2*.

Consider the following *elementary* dynamical system characterizing the evolution of the 2 variables $u_1 \equiv u_1(t), u_2 \equiv u_2(t),$ depending on the independent variable t ("time"):

$$\dot{u}_1 = u_2\ ,\ \dot{u}_2 = -u_1\ .$$

The general solution of this trivial system of 2 ODEs is

$$u_1(t) = A\sin(t+\theta),$$
$$u_2(t) = A\cos(t+\theta).$$

Let us now apply our standard invertible transformation relating $u_1 \equiv u_1(t), u_2 \equiv u_2(t)$ to $x_1 \equiv x_1(t), x_2 \equiv x_2(t)$ and viceversa. One thereby easily gets the following new dynamical system:

$$\dot{x}_1 = x_2 - F_2(x_1)$$
$$- F_1'\left(x_2 - F_2(x_1)\right)\left[x_1 - F_1\left(x_2 - F_2(x_1)\right)\right],$$

$$\dot{x}_2 = -x_1 + F_1\left(x_2 - F_2(x_1)\right)$$
$$+ F_2'(x_1)\left\{ x_2 - F_2(x_1) - F_1'\left(x_2 - F_2(x_1)\right) \cdot\right.$$
$$\left. \cdot \left[x_1 - F_1\left(x_2 - F_2(x_1)\right)\right] \right\}.$$

And the *general* solution of this dynamical system reads as follows:

$$x_1(t) = A\sin(t+\theta) + F_1\left(\mathrm{A}\cos(\mathrm{t}+\theta)\right),$$

$$x_2(t) = A\cos(t+\theta)$$
$$+F_2\left(A\sin(t+\theta)+F_1\left(\mathrm{A}\cos(\mathrm{t}+\theta)\right)\right).$$

Note that this implies that the dynamical system written above (with *arbitrary* $F_1(w), F_2(w)$) is *isochronous*: all its solutions are periodic with period $T = 2\pi$.

Another simple and instructive case is that obtained, in an analogous manner, from the elementary dynamical system

$$\dot{u}_1 = u_2 ,\ \dot{u}_2 = u_3 ,\ \dot{u}_3 = -\left(\rho\omega^2 u_1 + \omega^2 u_2 + \rho u_3\right) ,$$

the general solution of which reads

$$u_1(t) = A\sin(\omega t + \theta) + B\exp(-\rho t) ,$$
$$u_2(t) = A\omega\cos(\omega t + \theta) - B\rho\exp(-\rho t) ,$$
$$u_3(t) = -A\omega^2\sin(\omega t + \theta) + B\rho^2\exp(-\rho t) .$$

Here ω and ρ are two arbitrary constants, and if they are both real and ρ is positive, $\rho > 0$, then this general solution ---where A, B, θ are 3 arbitrary constants---is *asymptotically isochronous*, namely it becomes periodic in the remote future with the fixed period $T = 2\pi/\omega$, up to corrections vanishing exponentially, of order $\exp(-\rho t)$.

The corresponding system obtained via the simple invertible transformation

$$x_1 = u_1 + F_1(u_2, u_3),\ \ x_2 = u_2 + F_2(x_1, u_3),\ \ x_3 = u_3 + F_3(x_1, x_2)\ ,$$

$$u_3 = x_3 - F_3(x_1, x_2),\ \ u_2 = x_2 - F_2(x_1, u_3),\ \ u_1 = x_1 - F_1(u_2, u_3)\ ,$$

reads as follows:

$$\dot{x}_1 = u_2 + F_{1,1}(u_2, u_3) - F_{1,2}(u_2, u_3)\left(\rho\,\omega^2 u_1 + \omega^2 u_2 + \rho\, u_3\right),$$

$$\dot{x}_2 = u_3 + F_{2,1}(x_1, u_3)\,\dot{x}_1 - F_{2,1}(x_1, u_3)\left(\rho\,\omega^2 u_1 + \omega^2 u_2 + \rho\, u_3\right),$$

$$\dot{x}_3 = -\left(\rho\,\omega^2 u_1 + \omega^2 u_2 + \rho\, u_3\right) + F_{3,1}(x_1, x_2)\,\dot{x}_1 - F_{3,2}(x_1, x_2)\,\dot{x}_2\ ,$$

with

$$F_{n,k}(w_1, w_2) \equiv \frac{\partial F_n(w_1, w_2)}{\partial w_k},\ \ n = 1,2,3,\ \ k = 1,2\ ,$$

and u_1, u_2, u_3 given in terms of x_1, x_2, x_3 by the corresponding inverse transformation, see above.

And it is clear that the *general solution* of this system is *asymptotically isochronous*, indeed its explicit expression reads as follows:

$$x_1(t) = u_1(t) + F_1\big(u_2(t), u_3(t)\big),$$
$$x_2(t) = u_2(t) + F_2\big(x_1(t), u_3(t)\big),$$
$$x_3(t) = u_3(t) + F_3\big(x_1(t), x_2(t)\big),$$

with

$$u_1(t) = A\sin(\omega t + \theta) + B\exp(-\rho t),$$
$$u_2(t) = A\omega\cos(\omega t + \theta) - B\rho\exp(-\rho t),$$
$$u_3(t) = -A\omega^2\sin(\omega t + \theta) + B\rho^2\exp(-\rho t).$$

Hamiltonian systems

The class of Hamiltonian dynamical systems is characterized by the system of 2 ODEs

$$\dot{q} = \frac{\partial h(p,q)}{\partial p}, \quad \dot{p} = -\frac{\partial h(p,q)}{\partial q} ,$$

where the Hamiltonian function $h(p,q)$ is a priori arbitrary. Note that we are restricting here, for simplicity, attention to the case of a single canonical coordinate q and correspondingly a single canonical momentum. The fact to be highlighted is that, in this context, the standard invertible transformation---say, from $q \equiv u_1$ and $p \equiv u_2$ to new canonical variables $Q \equiv x_1$ and $P \equiv x_2$, hence reading

$$Q = q + F_1(p), \quad P = p + F_2(Q) = p + F_2\left(q + F_1(p)\right),$$

$$q = Q - F_1(p) = Q - F_1\left(P - F_2(Q)\right), \quad p = P - F_2(Q).$$

---is canonical, namely it leads to new equations of motion which retain the Hamiltonian form,

$$\dot{Q} = \frac{\partial H(P,Q)}{\partial P}, \quad \dot{P} = -\frac{\partial H(P,Q)}{\partial Q} ,$$

with

$$H(P,Q) = h\big(p(P,,Q), q(P,Q) \big) ,$$

where the functions $p(P,Q), q(P,Q)$ are of course provided by the standard transformation formulas indicated above, with the identification $q \equiv u_1$, $p \equiv u_2$ and $Q \equiv x_1$, $P \equiv x_2$ (or $Q \equiv x_2$, $P \equiv x_1$).

One can thereby manufacture, for instance, a multitude of *isochronous* Hamiltonian systems, such as

$$H(P,Q)=\frac{1}{2}\Big\{\left(P-\beta_0-\beta_1 Q-\beta_2 Q^2\right)^2$$
$$+\Big[\,Q-\alpha_0-\alpha_1\left(P-\beta_0-\beta_1 Q-\beta_2 Q^2\right)$$
$$-\alpha_2\left(P-\beta_0-\beta_1 Q-\beta_2 Q^2\right)^2\,\Big]^2\Big\},$$

which clearly obtains from the harmonic oscillator Hamiltonian $h(p,q)=\left(p^2+q^2\right)/2$ via the *canonical* transformation written above, and moreover with the assignments

$$F_1(w)=\alpha_0+\alpha_1 w+\alpha_2 w^2\,,\;\; F_2(w)=\beta_0+\beta_1 w+\beta_2 w^2\,.$$

Hence it features solutions which are *all periodic with period* 2π (and which can be easily written quite explicitly).

Remark. An interesting related issue is whether these Hamiltonians, after quantization, feature an equispaced spectrum. This may well depend on the specific prescription employed to make the transition from the classical to the quantal Hamiltonian. And to what extent are the corresponding stationary Schroedinger equations explicitly solvable?

Discrete-time **dynamical systems**

(in particular, *isochronous* and *asymptotically isochronous* examples)

Let us recall that a *discrete-time* dynamical system---or, equivalently, a multidimensional map---is characterized by a set of *N* "dependent variables" x_n, which are functions of a "discrete-time" independent variable ℓ taking (here and hereafter) nonnegative integer values, $x_n \equiv x_n(\ell), \ell = 0,1,2,...$, and which evolve in discrete time as follows:

$$\tilde{x}_n = f_n(\underline{x}),\ \ n = 1,...,N\ ;\ \ \underline{\tilde{x}} = \underline{f}(\underline{x}).$$

The notation here is analogous to that used above for standard (i. e., continuous-time) dynamical systems, except that now (and hereafter) superimposed tildes indicate that the independent variable has been advanced by one unit,

$$\tilde{x}_n \equiv \tilde{x}_n(\ell) \equiv x_n(\ell+1),\ \ n = 1,...,N\ .$$

A discrete-time dynamical system is called *isochronous* if there exists an *open, fully-dimensional* set of initial data $x_n(0)$ yielding solutions which are completely periodic with a fixed period L,

$$x_n(\ell+L)=x_n(\ell),\ \ n=1,...,N\ ;$$

$$\underline{x}(\ell+L)=\underline{x}(\ell),$$

where L is of course now a *fixed positive integer* (independent of the initial data). The isochronous discrete-time dynamical systems that we consider below are such that this relation holds for *arbitrary* initial data.

Several techniques to manufacture *isochronous continuous-time* dynamical systems are available, in addition to that described above, based on our invertible transformation. Hence a plethora of such *isochronous* models is known. But no analogous techniques are---to the best of my knowledge---available to manufacture *discrete-time isochronous* dynamical systems (see, however, my very recent paper entitled “The discrete-time goldfish”, submitted to J. Math. Phys. on September 18, 2010). The invertible transformations described above allow us to manufacture such systems by taking as starting point a *trivial discrete-time isochronous system*, in analogy to the procedure applied above in the context of *continuous-time* dynamical systems.

Again, in order to illustrate this approach in the simplest context, below we mainly restrict consideration to *two-dimensional* systems, which moreover allow neat graphical representations of their evolution, taking place in a plane. But we also discuss below---in analogy to what we did in the preceding section---a *tridimensional* example, displaying the possibility to manufacture *solvable discrete-time* systems that are *asymptotically isochronous*.

So, to begin with let us start from the following quite simple discrete-time 2-dimensional dynamical system:

$$\tilde{u}_1 = c\,u_1 - s\,u_2\,,\;\; \tilde{u}_2 = s\,u_1 + c\,u_2\;.$$

Here and throughout the discussion of discrete-time systems we use the short-hand notation

$$c \equiv \cos\left(\frac{2\pi}{\lambda}\right),\;\; s \equiv \sin\left(\frac{2\pi}{\lambda}\right),$$

so that, at every time step, the two-vector $\underline{u}(\ell)$ of which the 2 dependent variables $u_1(\ell)$ and $u_2(\ell)$ are the 2 components makes a counterclockwise rotation, by the angle $2\pi/\lambda$, in the $u_1 u_2$ Cartesian plane.

This evolution is obviously *solvable*, since clearly the two-vector $\underline{u}(\ell)$ obtains from the two-vector $\underline{u}(0)$ by performing a counterclockwise rotation, in the $u_1 u_2$ Cartesian plane, by the angle $2\pi\ell/\lambda$: hence the corresponding formulas expressing $u_1(\ell)$ and $u_2(\ell)$ in terms of the initial data $u_1(0)$ and $u_2(0)$ are simply

$$u_1(\ell) = \cos\left(\frac{2\pi\ell}{L}\right) u_1(0) - \sin\left(\frac{2\pi\ell}{L}\right) u_2(0),$$

$$u_2(\ell) = \sin\left(\frac{2\pi\ell}{L}\right) u_1(0) + \cos\left(\frac{2\pi\ell}{L}\right) u_2(0).$$

Clearly this evolution is *isochronous* with period *L* if $\lambda = L$ with *L* a *positive integer*. More generally, this evolution is as well *isochronous* with period *L* if λ is *rational*, $\lambda = L/M$ with *L* and *M* coprime integers (and, say, *L* positive).

Let us now perform our standard change of dependent variables, from the 2 variables $u_1(\ell)$ and $u_2(\ell)$ to the 2 variables $x_1(\ell)$ and $x_2(\ell)$. Then the discrete-time dynamical system characterizing the evolution of the 2 new dependent variables $x_1(\ell)$ and $x_2(\ell)$ reads as follows:

$$\tilde{x}_1 = c\,x_1 - s\,x_2 + s\,F_2(x_1) - c\,F_1(\,x_2 - F_2(x_1)\,)$$
$$+ F_1(\,s\,x_1 + c\,x_2 - c\,F_2(x_1) - s\,F_1(\,x_2 - F_2(x_1)\,)\,),$$

$$\tilde{x}_2 = s\,x_1 + c\,x_2 - c\,F_2(x_1) - s\,F_1(\,x_2 - F_2(x_1)\,) + F_2(\,\tilde{x}_1\,).$$

These 2 "equations of motion", constitute the new *discrete-time* dynamical system. This more complicated system is, via the invertible transformation, just as solvable as the original, trivial system, and obviously as well *isochronous* with period *L*, if $\lambda = L/M$ with *L* a positive integer and *M* a coprime integer. Clearly the freedom to choose arbitrarily the 2 functions $F_1(w)$ and $F_2(w)$ entails that this class of discrete dynamical systems is quite vast. But the multiple convolution of these functions that occurs in these equations of motion entails that these equations are generally not very simple.

Clearly the quantity

$$K = u_1^{\,2} + u_2^{\,2}$$

is a "constant of motion" for the evolution of the original system for the variables $u_1(\ell)$ and $u_2(\ell)$. Likewise the image of this constant under the transformation from the 2 variables $u_1(\ell)$ and $u_2(\ell)$ to the 2 variables $x_1(\ell)$ and $x_2(\ell)$ is a constant of motion for the new evolution of the 2 variables $x_1(\ell)$ and $x_2(\ell)$, reading

$$K = \left[x_1 - F_1\left(x_2 - F_2\left(x_1\right)\right)\right]^2 + \left[x_2 - F_2\left(x_1\right)\right]^2 .$$

Note that it depends on the assignment of the 2 functions $F_1(w)$ and $F_2(w)$, but not on the number λ which also characterizes the discrete-time dynamical system evolution.

The specific value of this constant *K* depends of course on the initial data $x_1(0)$ and $x_2(0)$. Plotting in the $x_1 x_2$ Cartesian plane for various values of *K* the curves defined by the formula written above---which might be quite complicated, but cannot feature any crossing---yields a qualitative assessment of the behavior of the dynamical system under consideration, each (discrete) trajectory of which is of course confined to lie on the curve characterized by the value of *K* determined by the initial data. Indeed, if λ is a large number, the fact that the discrete trajectory shall lie on the curve with the relevant value of *K*---as determined by the initial data---entails that the discrete evolution shall closely mimic a continuous evolution along that curve: of course the evolution of the *discrete-time* dynamical system shall be *periodic* if λ is *rational* and *nonperiodic* if λ is *irrational*, and in the latter case it will eventually seem to completely cover the relevant constant-*K* curve, although of course it shall, after a finite time ℓ, only cover ℓ different points on that curve. For small values of λ the *discrete* character of the evolution may instead have a more dramatic connotation, with the moving point characterized by the Cartesian coordinates $x_1(\ell)$ and $x_2(\ell)$ jumping sequentially from one point of the constant-*K* curve to another point on that curve possibly quite far from the previous one. Of course this phenomenology shall be particularly striking in the case of a *rational* value of $\lambda = L/M$ with *L* a *small* positive integer (and *M* a coprime integer).

Example. Let us make the simple assignments

$$F_1(w)=\alpha w^2 , \; F_2(w)=\beta w^2 \; ,$$

where α and β are 2 *a priori* arbitrary numbers. Then the new discrete-time dynamical system reads as follows:

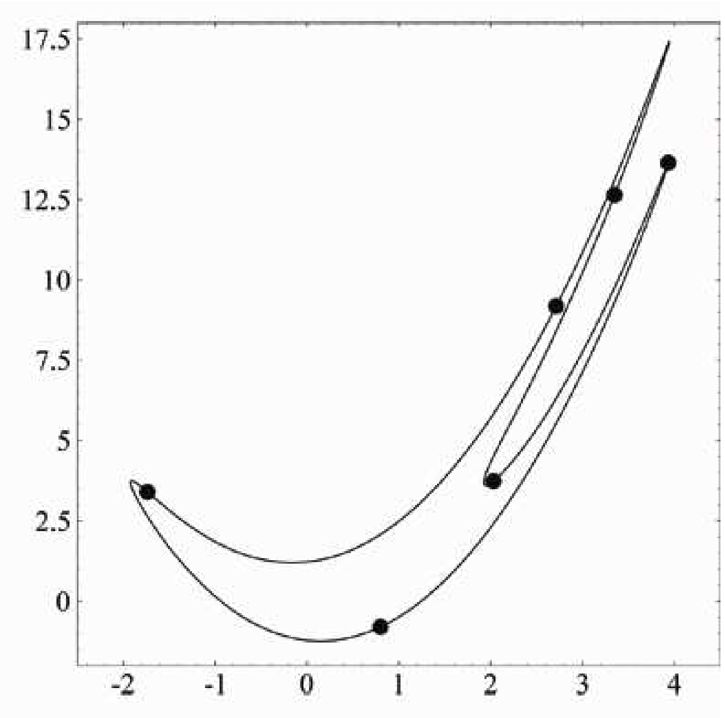
$$\tilde{x}_1 = c\left[x_1-\alpha\left(x_2-\beta x_1^{\;2}\right)^2\right]-s\left(x_2-\beta x_1^{\;2}\right)$$
$$+\alpha\left\{s\left[x_1-\alpha\left(x_2-\beta x_1^{\;2}\right)^2\right]+c\left(x_2-\beta x_1^{\;2}\right)\right\}^2 \; ,$$

$$\tilde{x}_2 = s\left[x_1-\alpha\left(x_2-\beta x_1^{\;2}\right)^2\right]+c\left(x_2-\beta x_1^{\;2}\right)+\beta \tilde{x}_1^{\;2} \; .$$

This discrete-time dynamical system is fairly complicated: indeed the right-hand side of the first equation features a term with the dependent variable x_1 raised to the 8-th power, and the right-hand side of the second equation features a term with the dependent variable x_1 raised to the 16-th power. In this case the constant of motion K reads

$$K = \left[x_1 - \alpha\left(x_2 - \beta x_1^{\,2}\right)^2\right]^2 + \left(x_2 - \beta x_1^{\,2}\right)^2 .$$

The dynamical evolution entailed by this model is fairly rich, while being solvable, and of course *isochronous* whenever λ is an *integer* (or a *rational* number). See the Figure for a display of the trajectory in the $x_1 x_2$ Cartesian plane of this system (with $\alpha = \beta = 1, \lambda = 6, x_1(0) = 0.8, x_2(0) = -0.8$ implying *K*=3.69566.



Let us also mention that, if the initial values assigned are somewhat larger than those of the trajectory reported in this Figure, the corresponding trajectory can reach quite large values before returning to the initial values: for instance the initial values $x_1(0) = 3, x_2(0) = 3$ yield $x_1(3) = 69, x_2(3) = 4767$, with *K*=1125.

Let us also report a discrete-time dynamical system involving 3 dependent variables and displaying the *asymptotically isochronous* phenomenology (see the somewhat analogous continuous-time dynamical system discussed above).

An asymptotically isochronous discrete-time system involving 3 dependent variables. Let

$$\tilde{u}_1 = u_2\,,\ \tilde{u}_2 = u_3\,,\ \tilde{u}_3 = \alpha u_1 - (1 + 2\alpha c) u_2 + (\alpha + 2c) u_3\ ,$$

where α and λ are two (a priori arbitrary) real numbers.

It is easily seen that the *general* solution of this discrete-time model reads as follows:

$$u_1(\ell) = A\alpha^{\ell} + B\cos\left(\frac{2\pi\ell}{L}\right) + C\sin\left(\frac{2\pi\ell}{L}\right),$$

$$u_2(\ell) = u_1(\ell+1),\ u_3(\ell) = u_2(\ell+1) = u_1(\ell+2)\,,$$

with A, B, C three arbitrary constants (of course easily expressible in terms of the initial data).

Hence the solution is, for arbitrary initial data, *isochronous* with period L (i. e., $u_n(\ell+L)=u_n(\ell)$) if λ is *rational*, $\lambda=L/M$ with L and M coprime integers (and, say, L positive) and moreover $\alpha=1$ (if $\alpha=-1$, it is as well *isochronous*, with period L if L is even and 2L if L is odd), and it is *asymptotically isochronous* (i. e., $u_n(\ell+L)-u_n(\ell)=O\left(|\alpha|^{\ell}\right)$ as $\ell\to\infty$) if $\lambda=L/M$ and $|\alpha|<1$.

A class of apparently quite less trivial discrete-time dynamical systems is then obtained via the standard invertible change of variables, from u_1,u_2,u_3 to x_1,x_2,x_3 and viceversa:

$$x_1=u_1+F_1(u_2,u_3),\ \ x_2=u_2+F_2(x_1,u_3),\ \ x_3=u_3+F_3(x_1,x_2);$$
$$u_3=x_3-F_3(x_1,x_2),\ \ u_2=x_2-F_2(x_1,u_3),\ \ u_1=x_1-F_1(u_2,u_3).$$

It involves the 3 arbitrary functions $F_n(w_1, w_2),\ n=1,2,3$, and it reads:

$$\tilde{x}_1 = x_2 - G_2(x_1, x_2, x_3) + G_4(x_1, x_2, x_3),$$

$$\tilde{x}_2 = x_3 - G_1(x_1, x_2, x_3) + G_5(x_1, x_2, x_3),$$

$$\tilde{x}_3 = J(x_1, x_2, x_3) + F_3(\tilde{x}_1, \tilde{x}_2),$$

where the 5 functions G_m are recursively given, one in terms of the other, as follows:

$$G_1(x_1, x_2, x_3) = F_3(x_1, x_2),$$

$$G_2(x_1, x_2, x_3) = F_2(x_1, x_3 - G_1(x_1, x_2, x_3)),$$

$$G_3(x_1, x_2, x_3) = F_1(x_2 - G_2(x_1, x_2, x_3), x_3 - G_1(x_1, x_2, x_3)),$$

$$G_4(x_1, x_2, x_3) = F_1(x_3 - G_1(x_1, x_2, x_3), J(x_1, x_2, x_3)),$$

$$G_5(x_1, x_2, x_3) = F_2(x_2 - G_2(x_1, x_2, x_3) + G_4(x_1, x_2, x_3), J(x_1, x_2, x_3)),$$

and the function *J* depends on G_1, G_2, G_3 as follows:

$$J(x_1, x_2, x_3) = \alpha\left[x_1 - G_3(x_1, x_2, x_3)\right]$$
$$+(1+2\alpha c)\left[x_2 - G_2(x_1, x_2, x_3)\right]$$
$$+(\alpha+2c)\left[x_3 - G_1(x_1, x_2, x_3)\right].$$

The general solution of this discrete-time model reads as follows:

$$x_1(\ell) = u_1(\ell) + F_1\big(u_1(\ell+1), u_1(\ell+2)\big),$$

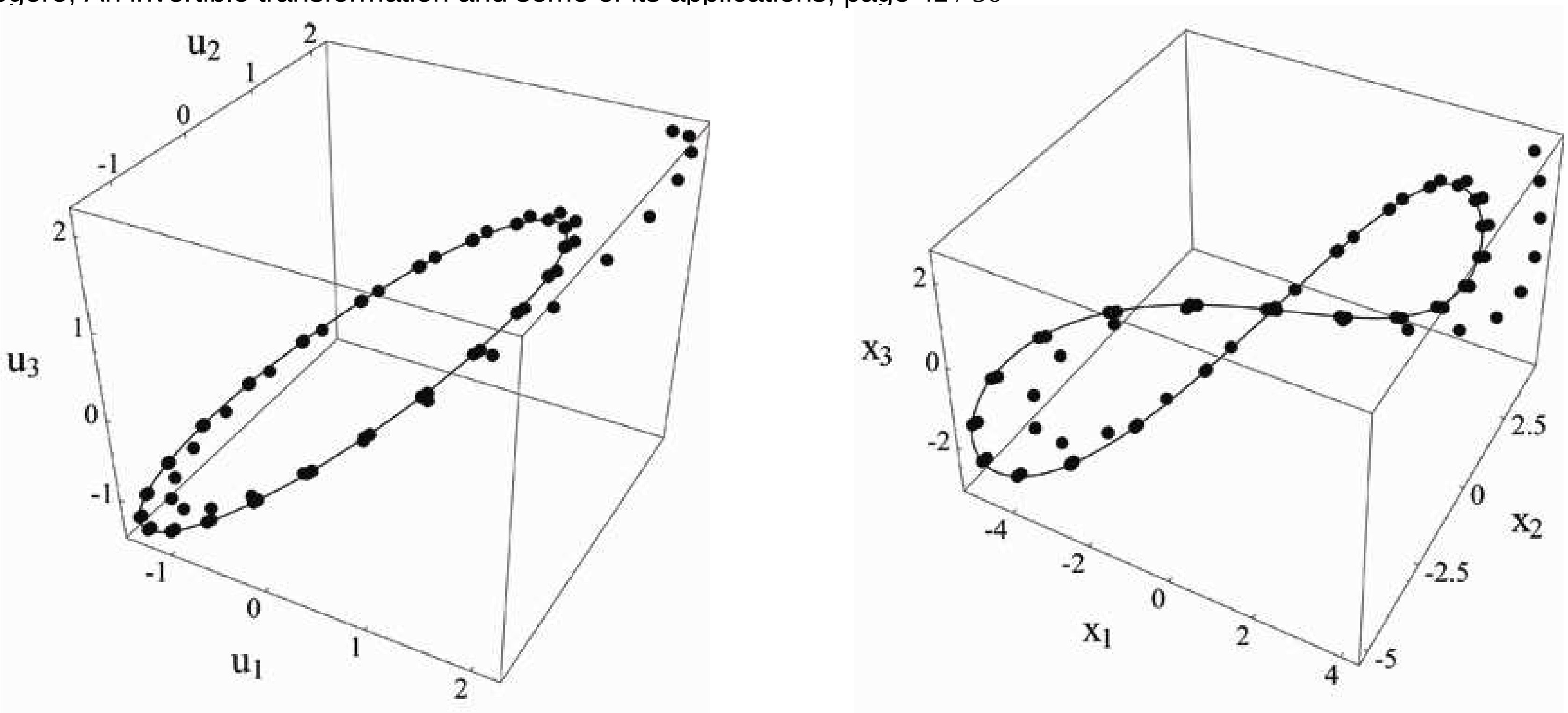
$$x_2(\ell) = u_1(\ell+1) + F_2\big(x_1(\ell), u_1(\ell+2)\big),$$

$$x_3(\ell) = u_1(\ell+2) + F_3\big(x_1(\ell), x_2(\ell)\big),$$

With

$$u_1(\ell) = A\alpha^{\ell} + B\cos\left(\frac{2\pi\ell}{L}\right) + C\sin\left(\frac{2\pi\ell}{L}\right).$$

And clearly this system inherits the same phenomenology---concerning *isochrony* and *asymptotic isochrony*---of the simple linear model to which it is related via the invertible transformation, as described above.



The Figure displays the orbit and limit trajectory in the $u_1 u_2 u_3$ Cartesian space for the original system and the corresponding orbit and limit trajectory in the $x_1 x_2 x_3$ Cartesian space for the system obtained via the transformation reported above with

$$F_1(p,q) = p + q - \frac{p\,q}{2},\;\; F_2(p,q) = p - q,\;\; F_3(p,q) = p^2 - q^2 \;.$$

Solvable systems of autonomous nonlinear PDEs

Here we start again from a trivially *solvable* model, and via an invertible transformation we obtain a new model, also *solvable*, which looks much less trivial. For simplicity the treatment is restricted to a system of 2 first-order PDEs featuring 2 dependent and 2 independent variables.

We take as point of departure the following trivial system of 2 linear PDEs,

$$\varphi_{1,t} = \varphi_{2,x} \,, \;\; \varphi_{2,t} = \varphi_{1,x} \;,$$

where the 2 functions $\varphi_n \equiv \varphi_n(x,t)$ depend on the 2 independent variables x and t. Here and below subscripted variables indicate partial differentiation with respect to them. Clearly this system of 2 linear PDEs has the following *general* solution:

$$\varphi_1(x,t) = \Phi_+(x+t) \equiv \Phi_1(x+t) + \Phi_2(x-t),$$
$$\varphi_2(x,t) = \Phi_-(x+t) \equiv \Phi_1(x+t) - \Phi_2(x-t),$$

where $\Phi_n(z)$ are 2 *arbitrary* functions of the single variable z.

We now apply the following invertible transformation:

$$\psi_1=\frac{\varphi_1\ f_{11}(\varphi_2)+f_{12}(\varphi_2)}{\varphi_1\ f_{13}(\varphi_2)+f_{14}(\varphi_2)},\ \psi_2=\frac{\varphi_2\ f_{21}(\psi_1)+f_{22}(\psi_1)}{\varphi_2\ f_{23}(\psi_1)+f_{24}(\psi_1)},$$

$$\varphi_1=-\frac{\psi_1\ f_{14}(\varphi_2)-f_{12}(\varphi_2)}{\psi_1\ f_{13}(\varphi_2)-f_{11}(\varphi_2)},\ \varphi_2=-\frac{\psi_2\ f_{24}(\psi_1)-f_{22}(\psi_1)}{\psi_2\ f_{23}(\psi_1)-f_{21}(\psi_1)}.$$

These *explicit* transformations involve 8 *a priori arbitrary* functions $f_{nk}(w)$. However we now restrict attention to a very specific example (due to Matteo Sommacal), corresponding to the following assignment:

$$f_{11}(w)=f_{14}(w)=\frac{c_4+c_1\,c_3+w(c_3+c_1\,c_4)}{c_3^{\ 2}-c_4^{\ 2}},$$

$$f_{12}(w)=f_{13}(w)=-\frac{c_2\,(c_3+w\,c_4)}{c_3^{\ 2}-c_4^{\ 2}},$$

$$f_{21}(w)=f_{24}(w)=c_3\,w,\ \ f_{22}(w)=f_{23}(w)=c_4\,w.$$

Then the two functions $\psi_n \equiv \psi_n(x,t)$ satisfy the following system of two coupled nonlinear PDEs:

$$\psi_{1,t} = \left[\alpha\,\psi_{1,x} + \left(\beta^2 - \alpha^2\right)\psi_{2,x}\right]/\,\beta\,,$$

$$\psi_{2,t} = \left(\psi_{1,x} - \alpha\,\psi_{2,x}\right)/\,\beta\,,$$

where

$$\alpha \equiv \alpha(\psi_1,\psi_2) = \frac{c_2\left(1-\psi_1{}^2\right)}{\left(c_1+c_2+\psi_2\right)\left(c_1-c_2+\psi_2\right)},$$

$$\beta \equiv \beta(\psi_1,\psi_2) = \frac{\left(c_3{}^2-c_4{}^2\right)\left(c_1+c_2\,\psi_1+\psi_2\right)}{\left(c_1+c_2+\psi_2\right)\left(c_1-c_2+\psi_2\right)\left(c_3-c_4\,\psi_2\right)^2}\,.$$

The *general* solution $\psi_1=\psi_1(x,t),\psi_2=\psi_2(x,t)$ of this system of two PDEs then reads as follows:

$$\psi_1=\frac{c_2\,c_3-(c_1\,c_3+c_4)\Phi_+ +c_2\,c_4\,\Phi_- -(c_1\,c_4+c_3)\Phi_+\,\Phi_-}{-c_1\,c_3-c\;\;+c_1\,c_3\,\Phi_+ -(c_1\,c_4+c_3)\Phi_- +c_2\,c_4\,\Phi_+\,\Phi_-},$$

$$\psi_2=\frac{c_4+c_3\,\Phi_-}{c_3+c_4\,\Phi_-}\ ,$$

where $\Phi_\pm=\Phi_\pm(x,t)$ are of course defined as above,

$$\Phi_+(x+t)\equiv\Phi_1(x+t)+\Phi_2(x-t),$$

$$\Phi_-(x+t)\equiv\Phi_1(x+t)-\Phi_2(x-t),$$

in terms of the 2 *arbitrary* functions $\Phi_n(z)$ of the single variable z .

A *solvable* nonautonomous PDE

Here we show---as a representative example---how to manufacture a *solvable* nonautonomous PDE starting from the trivial (linear) autonomous first-order PDE

$$\varphi_u(u,w)=\varphi_w(u,w)\,,$$

the *general* solution of which reads of course

$$\varphi(u,w)=F(u+w)$$

where $F(z)$ is an *arbitrary* function of the single variable z. We set

$$\varphi(u,w)=\psi(x,y)$$

with

$$x=\frac{u\ f_{11}(w)+\ f_{12}(w)}{u\ f_{13}(w)+\ f_{14}(w)},\ \ y=\frac{w\ f_{21}(x)+\ f_{22}(x)}{w\ f_{23}(x)+\ f_{24}(x)},$$

$$u=-\frac{x\ f_{14}(w)-\ f_{12}(w)}{x\ f_{13}(w)-\ f_{11}(w)},\ \ w=-\frac{y\ f_{24}(x)-\ f_{22}(x)}{y\ f_{23}(x)-\ f_{21}(x)}.$$

It is then a matter of trivial if tedious algebra to ascertain that $\psi(x,y)$ satisfies the following (linear) nonautonomous PDE:

$$g(x,y)\psi_x(x,y)=h(x,y)\psi_y(x,y)$$

with $g(x,y)$ and $h(x,y)$ expressed explicitly (by rather complicated formulas not reported here) in terms of the 8 arbitrary functions $f_{nk}(z), n=1,2, k=1,2,3,4.$
We limit our presentation to exhibit here one specific example (due to François Leyvraz), corresponding to the assignments

$$f_{11}(z)=z,\;\; f_{12}(z)=f_{13}(z)=0,\;\; f_{14}(z)=1,$$
$$f_{21}(z)=f_{24}(z)=\cos(z),\; f_{13}(z)=-f_{22}(z)=\sin(z)\,.$$

Then the linear nonautonomous PDE satisfied by $\psi(x,y)$ reads as follows:

$$\begin{aligned}
&(y\cos x+\sin x)^2(\cos x-y\sin x)\cdot\\
&\cdot\left[3\left(1+y^2\right)+\left(1-y^2\right)\cos(2x)-2y\sin(2x)\right]\psi_x(x,y)\\
&=2\left[(\cos x-y\sin x)\cos\left(\frac{y\cos x+\sin x}{\cos x-y\sin x}\right)+(y\cos x+\sin x)\sin x\right]^2\cdot\\
&\cdot\left[(1+x)(y\cos x+\sin x)^2-x\left(1+y^2\right)\right]\psi_y(x,y)\,.
\end{aligned}$$

And its *general* solution reads as follows:

$$\psi(x,y)=F\left(\frac{x\cos x-x\,y\sin x}{y\cos x+\sin x}+\frac{y\cos x+\sin x}{\cos x-y\sin x}\right),$$

with $F(z)$ an *arbitrary* function.

On the face of it, the fact that the PDE written above is explicitly *solvable* should appear quite nontrivial to anybody who does not know how this finding has been arrived at; although verifying it is a relatively trivial task.

Functional equations

Here we report an, apparently nontrivial, functional equation involving 2 functions, as an example of the kind of findings obtainable via this approach. It reads as follows:

$$x_1(z_1+z_2)=\frac{u_1(z_1)u_1(z_2)\,f_{11}(u_1(z_1)+u_1(z_2))+f_{11}(u_1(z_1)+u_1(z_2))}{u_1(z_1)u_1(z_2)\,f_{13}(u_1(z_1)+u_1(z_2))+f_{14}(u_1(z_1)+u_1(z_2))},$$

$$x_2(z_1+z_2)=\frac{[u_2(z_1)+u_2(z_2)]\,f_{21}(x_1(z_1+z_2))+f_{22}(x_1(z_1+z_2))}{[u_2(z_1)+u_2(z_2)]\,f_{23}(x_1(z_1+z_2))+f_{24}(x_1(z_1+z_2))},$$

where, in the 2 preceding formulas, firstly $u_1(z)$ should be replaced by the following expression in terms of $x_1(z)$ and $u_2(z)$,

$$u_1(z) = -\frac{x_1(z)\, f_{14}(u_2(z)) - f_{12}(u_2(z))}{x_1(z)\, f_{13}(u_2(z)) - f_{11}(u_2(z))},$$

and subsequently $u_2(z)$ should be replaced by the following expression in terms of $x_1(z)$ and $x_2(z)$,

$$u_2(z) = -\frac{x_2(z)\, f_{24}(x_1(z)) - f_{22}(x_1(z))}{x_2(z)\, f_{23}(x_1(z)) - f_{21}(x_1(z))},$$

so that the resulting formulas relate (explicitly, if in a convoluted manner) the values that the two functions $x_1(z)$ and $x_2(z)$ take at the value $z = z_1 + z_2$ of their argument, to the values they take at z_1 and at z_2 (where z_1 and z_2 are of course independent variables taking *arbitrary* values).

The (explicit !) solution of this functional equation reads as follows:

$$x_1(z)=\frac{\exp(b\,z)\,f_{11}(a\,z)+f_{12}(a\,z)}{\exp(b\,z)\,f_{13}(a\,z)+f_{14}(a\,z)}\ ,$$

$$x_2(z)=\Big[\ a\,z\,f_{21}\left(\frac{\exp(b\,z)\,f_{11}(a\,z)+f_{12}(a\,z)}{\exp(b\,z)\,f_{13}(a\,z)+f_{14}(a\,z)}\right)$$

$$+\,f_{22}\left(\frac{\exp(b\,z)\,f_{11}(a\,z)+f_{12}(a\,z)}{\exp(b\,z)\,f_{13}(a\,z)+f_{14}(a\,z)}\right)\ \Big]\cdot$$

$$\Big[\ a\,z\,f_{23}\left(\frac{\exp(b\,z)\,f_{11}(a\,z)+f_{12}(a\,z)}{\exp(b\,z)\,f_{13}(a\,z)+f_{14}(a\,z)}\right)$$

$$+\,f_{24}\left(\frac{\exp(b\,z)\,f_{11}(a\,z)+f_{12}(a\,z)}{\exp(b\,z)\,f_{13}(a\,z)+f_{14}(a\,z)}\right)\ \Big]^{-1}\ ,$$

where a,b are two *arbitrary* parameters. And note that the 8 functions $f_{nk}(w),\ n=1,2,\ k=1,2,3,4$ appearing in the above functional equation and in its solution are *arbitrary*.

This finding clearly obtains, via the transformations

$$x_1 = \frac{u_1\ f_{11}(u_2) + f_{12}(u_2)}{u_1\ f_{13}(u_2) + f_{14}(u_2)},\ \ x_2 = \frac{u_2\ f_{21}(x_1) + f_{22}(x_1)}{u_2\ f_{23}(x_1) + f_{24}(x_1)},$$

$$u_1 = -\frac{x_1\ f_{14}(u_2) - f_{12}(u_2)}{x_1\ f_{13}(u_2) - f_{11}(u_2)},\ \ u_2 = -\frac{x_2\ f_{24}(x_1) - f_{22}(x_1)}{x_2\ f_{23}(x_1) - f_{21}(x_1)},$$

from the two trivial functional equations

$$u_1(z_1 + z_2) = u_1(z_1)u_1(z_2),\ \ u_2(z_1 + z_2) = u_2(z_1) + u_2(z_2)\ ,$$

whose solutions of course read

$$u_1(z) = \exp(b\,z),\ \ u_2(z) = a\,z\ .$$

Analytical geometry

The standard invertible transformation

$$x_1 = u_1 + F_1(u_2),\ \ x_2 = u_2 + F_2(x_1) = u_2 + F_2\left(u_1 + F_1(u_2)\right),$$
$$u_1 = x_1 - F_1(u_2) = x_1 - F_1\left(x_2 - F_2(x_1)\right),\ \ u_2 = x_2 - F_2(x_1)$$

is an *area preserving* reparametrization of the Cartesian plane: indeed we already mentioned that---as it is indeed easy to verify---it entails that the Jacobian determinant of this change of variables---from u_1, u_2 to x_1, x_2 and viceversa---is *unity*, for any arbitrary assignment of the 2 functions $F_1(w), F_2(w)$ (this property corresponds---in the Hamiltonian context, see Section 4.3.1---to the canonical character of the corresponding change of variables).

A representative example of the findings that easily flow from this property is provided by the following *Proposition*.

Proposition: Let c_1, c_2 be 2 arbitrary (real) numbers, and draw, in the $x_1 \, x_2$ Cartesian plane, the following 4 curves: the curve A going from the point $a=(0,0)$ to the point $b=(1,c_2)$ and characterized by the equation (a piece of a parabola)

$$A: \; x_2 = c_2 \, x_1^{\;2} \; ;$$

the curve B going from the point $b=(1,c_2)$ to the point $c=(c_1, 1+c_1^{\;2} \, c_2)$ and characterized by the (quartic) equation

$$B: \; 1 - x_1 - x_2 + c_2 \, x_1^{\;2} + c_1 \, x_2^{\;2} - 2 \, c_1 \, c_2 \, x_1^{\;2} \, x_2 + c_1 \, c_2^{\;2} x_1^{\;4} = 0 \; ;$$

the curve C going from the point $c=(c_1, 1+c_1^{\;2} \, c_2)$ to the point $a=(0,0)$ and characterized by the (quartic) equation

$$C: \; x_1 - c_1 \, x_2^{\;2} + 2 \, c_1 \, c_2 \, x_1^{\;2} \, x_2 - c_1 \, c_2^{\;2} x_1^{\;4} = 0 \; ;$$

and the curve D going from the point $a=(0,0)$ to the point $c=\left(\frac{1}{2}+\frac{c_1}{4}, \frac{1}{2}+c_2\left(\frac{1}{2}+\frac{c_1}{4}\right)^2\right)$

(lying on the curve B) and characterized by the (quartic) equation

$$D: x_1 - x_2 + c_2 \, x_1^{\;2} - c_1 \, x_2^{\;2} + 2 c_1 \, c_2 \, x_1^{\;2} \, x_2 - c_1 \, c_2^{\;2} x_1^{\;4} = 0 .$$

Then the region enclosed by the 3 curves A, B, C has area 1/2, and the curve D divides this region in two parts of equal area (see Figure). □

The proof of this *Proposition* is an immediate consequence of the fact that, via the above transformation with $F_n(w)=c_n\,w^2$ the region enclosed by the 3 curves A,B,C corresponds, in the $x_1\,x_2$ Cartesian plane, to the triangle of vertices $(0,0),(1,0),(0,1)$ in the $u_1\,u_2$ Cartesian plane, and likewise the curve D corresponds to the segment starting from the vertex $a=(0,0)$ and bisecting that triangle (see Figure).