Isochronous dynamical systems, the arrow of time and the definitions of “chaotic” versus “integrable” behaviors

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Summary

Any (autonomous) dynamical system can be *extended* or *modified*, obtaining thereby a *new* (autonomous) dynamical system involving a constant *T*---the value of which can be freely assigned---and featuring the following two properties: (i) all solutions of the new model are *isochronous* (completely periodic in all their degrees of freedom with the assigned period *T*); (ii) starting from *generic* initial data, the time evolution of the new dynamical system over time intervals of order $\tilde{T} << T$ is *essentially identical* to that of the original dynamical system, up to a *constant* rescaling of time and of corrections of order $\tilde{T}/T$. These findings entail that, in some sense, “*isochronous systems are not rare*” and moreover that such systems may feature an “*extremely complicated* “ time-evolution. They are also valid in the context of *Hamiltonian* dynamics; they are in particular applicable to the most general many-body problem (provided it is, overall, translation-invariant), entailing remarkable observations about statistical mechanics, thermodynamics and the issue of the “arrow of time” for macroscopic physics. Since *completely periodic* systems are *maximally superintegrable* (possessing the maximal number of functionally independent constants of motion compatible with the time evolution not being frozen), these findings also entail that *any* (Hamiltonian) dynamics can be *embedded* into a *superintegrable* (Hamiltonian) dynamics; and again, that such *superintegrable* systems may feature an “*extremely complicated* “ time-evolution.

All these findings have been obtained together with **François Leyvraz**. Some of them are reported in a recent monograph (F. Calogero, *Isochronous systems*, Oxford University Press, 2008); others are more recent, see references listed below. A very recent finding demonstrates how to extend an *arbitrary* (autonomous) dynamical system so that the (also autonomous) extended system is *isochronous* (with an *arbitrarily assigned* period *T*) yet its dynamics for an *arbitrary* fraction (of course, less than unity) of its (periodic) time evolution is *exactly identical* to that of the original system.

These findings suggest the need to invent new definitions **associated with a *finite* time scale** of “chaotic” versus “integrable” behaviors of dynamical systems (all current definitions refer instead to the behavior over *infinite* time).

Main references

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A trick to transform a dynamical system into an *isochronous* dynamical system

Let us start from an *autonomous*, but otherwise largely arbitrary, dynamical system, say

$$\underline{X}'=\underline{h}\left(\underline{X}\right);\;\; X_n'=h_n\left(\underline{X}\right),\;\; n=1,...,N\,,\;\; (*)$$

with the appended prime indicating differentiation with respect to the variable τ of $\underline{X}(\tau)$.

We then *change* it so that it reads as follows:

$$\dot{\underline{x}}=\dot{\tau}(t)\,\underline{h}\left(\underline{x}\right);\;\; \dot{x}_n=\dot{\tau}(t)\,h_n\left(\underline{x}\right),\;\; n=1,...,N\,,\;\; (**)$$

with the superimposed dot indicating differentiation with respect to the variable *t* of $\underline{x}(t)$ and $\tau(t)$. .

It is then plain that the general solution of this system reads

$$\underline{x}(t)=\underline{X}\left[\tau(t)\right];\;\; x_n(t)=X_n\left[\tau(t)\right],\;\; n=1,...,N\,,$$

where $\underline{X}(\tau)$ is the general solution of (*). Hence if the scalar function $\tau(t)$ is periodic with period *T*,

$$\tau(t+T)=\tau(t),$$

the general solution of (**) *inherits* from $\tau(t)$ the property to be *isochronous* with period *T*:

$$\underline{x}(t+T)=\underline{x}(t).$$

In conclusion, whenever $\underline{h}(\underline{X})$ is such that the solution of the system (*) of *N* nonlinear ODEs exists globally, *all* solutions of the system (**) are periodic with period *T*: this system is ***isochronous***.

However, the system (**) is *not* autonomous. To eliminate this "defect", we perform the second step of our treatment, replacing this system (**) with the system

$$\dot{\underline{x}} = \phi\, \underline{h}(\underline{x});\;\; \dot{x}_n = \phi\, h_n(\underline{x}),\;\; n = 1,\ldots,N,\;\; (***)$$

which is of course equivalent provided the time-evolution of the scalar quantity ϕ is such that

$$\phi(t) = \dot{\tau}(t).$$

There are now two options to obtain a quantity ϕ that qualifies for this purpose. One option---perhaps the most obvious---is to treat ϕ as an additional dependent variable, and to *extend* the system (*) by attaching to it a few additional ODEs involving ϕ and possibly other, additional dependent variables, so as to guarantee that the time evolution of ϕ has the desired property. Specific instances of how to achieve this goal are detailed in a paper [F. Calogero and F. Leyvraz, “How to extend any dynamical system so that it becomes isochronous, asymptotically isochronous or multi-periodic”, J. Nonlinear Math. Phys. **16**, 311-338 (2009)], where it is also shown how a third step of our treatment---consisting essentially in a change of the additional dependent variables, entangling them with the original variables---allows to manufacture many, quite neat, extended dynamical systems having the required properties of *isochrony*. One example obtained in this manner is displayed below.

Another option is to identify a collective variable $\phi(\underline{x})$ that, as a consequence of the very evolution entailed by the dynamical system (***), has a time evolution, $\phi(t) \equiv \phi[\underline{x}(t)]$, such that, via the preceding formula, it defines a function $\tau(t)$ having the desired properties. This is the approach on which we will then focus, restricting moreover attention to a Hamiltonian system of major physical relevance.

Example: a modified Lorenz system

$$\dot{x}_1 = -\alpha\, y_1 \left(1 + x_1^2\right)\left(x_1 - x_2\right),$$

$$\dot{x}_2 = y_1 \left(1 + x_1^2\right)\left(\beta\, x_1 - x_2 - x_1\, x_3\right),$$

$$\dot{x}_3 = y_1 \left(1 + x_1^2\right)\left(x_1\, x_2 - \gamma\, x_3\right),$$

$$\dot{y}_1 = \Omega\, y_2 + 2\alpha\, x_1 \left(x_1 - x_2\right) y_1^{\;2},$$

$$\dot{y}_2 = -\Omega\, y_1 + 2\alpha\, x_1 \left(x_1 - x_2\right) y_1\, y_2 .$$

For $\Omega = 0$ one can ignore y_2 (whose evolution does not influence the other variables) and set $y_1\left(1 + x_1^2\right) = 1$: then the fourth of these 5 ODEs coincides with the first, and the first 3 become the 3 ODEs of the Lorenz model:

$$X_1' = -\alpha\,\left(X_1 - X_2\right),$$

$$X_2' = \beta\, X_1 - X_2 - X_1 X_3\ ,$$

$$X_3' = X_1 X_2 - \gamma X_3\ .$$

Here the appended prime indicates differentiation with respect to the independent variable, which for convenience we call τ, so that

$$X_n \equiv X_n(\tau),\quad X_n' \equiv X_n'(\tau) \equiv dX_n(\tau)/d\tau\ .$$

But in fact also the solution of our *isochronous* model --- with $\Omega > 0$ --- can be “explicitly” exhibited in terms of the solutions $X_n(\tau)$ of the Lorenz model, with the same initial data,

$$X_n(0) = x_n(0)\ :$$

$$x_n(t) = X_n[\tau(t)], \quad n = 1,2,3,$$

$$y_1(t) = \dot{\tau}(t) / \left\{1 + [x_1(t)]^2\right\},$$

$$y_2(t) = \Omega^{-1} \ddot{\tau}(t) / \left\{1 + [x_1(t)]^2\right\},$$

$$\tau(t) = \{A \sin(\Omega t) + B[1 - \cos(\Omega t)]\} / \Omega \ .$$

This solution is clearly periodic with period

$$T = 2\pi / \Omega \ .$$

And clearly, for $0 < t << T$,

$$\tau(t) = At + O(t/T) .$$

A trick to *modify* a Hamiltonian into an *isochronous* Hamiltonian

$$\left[\ H\left(\underline{p},\underline{q}\right),\Theta\left(\underline{p},\underline{q}\right)\ \right]=1$$

Isochronous Hamiltonian:

$$\tilde{H}\left(\underline{p},\underline{q};\Omega\right)=\frac{1}{2}\left\{\left[H\left(\underline{p},\underline{q}\right)\right]^2+\Omega^2\left[\Theta\left(\underline{p},\underline{q}\right)\right]^2\right\};\quad , \quad T=\frac{2\pi}{\Omega}$$

“Isochronous Hamiltonian systems are not rare”

For a proof see: F. Calogero and F. Leyvraz, “General technique to produce isochronous Hamiltonians”, J. Phys. A.: Math. Theor. **40**, 12931-12944 (2007).

A remarkable example: the many-body problem

We write as follows the (simplest version of the) Hamiltonian characterizing the standard nonrelativistic *N*-body problem:

$$H(\underline{p},\underline{q})=\frac{1}{2}\sum_{n=1}^{N}\left[p_n^2\right]+V(\underline{q}),\ V(\underline{q}+a)=V(\underline{q}).$$

Let us now review some standard related developments, trivial as they are.

We hereafter denote with P the total momentum, and with Q the (canonically-conjugate) centre-of-mass coordinate:

$$P=\sum_{n=1}^{N} p_n\,,\ Q=\frac{1}{N}\sum_{n=1}^{N} q_n\,.$$

Thanks to the translation invariance property

$$[H,P]=0$$

Here and hereafter the Poisson bracket $[F,G]$ of two functions $F(\underline{p},\underline{q})$ and $G(\underline{p},\underline{q})$ of the canonical variables is defined as follows:

$$[F,G]=\sum_{n=1}^{N}\left[\frac{\partial F(\underline{p},\underline{q})}{\partial p_n}\frac{\partial G(\underline{p},\underline{q})}{\partial q_n}-\frac{\partial G(\underline{p},\underline{q})}{\partial p_n}\frac{\partial F(\underline{p},\underline{q})}{\partial q_n}\right].$$

And let us recall that the evolution of any function $F(\underline{p},\underline{q})$ of the canonical coordinates is determined by the equation

$$F'=[H,F],$$

where the appended prime denotes differentiation with respect to the "timelike" variable corresponding to the evolution induced by the Hamiltonian *H*.

It is now convenient to introduce the "relative coordinates" x_n and the "relative momenta" y_n via the standard definitions

$$x_n = q_n - Q, \; y_n = p_n - \frac{P}{N} .$$

Note that these are not canonically conjugated quantities, since $[y_n, x_m] = \delta_{nm} - 1/N$, and they are not independent since obviously their sum vanishes:

$$\sum_{n=1}^{N} y_n = 0, \; \sum_{n=1}^{N} x_n = 0.$$

It is moreover convenient to introduce the "relative-motion" Hamiltonian $h(\underline{y}, \underline{x})$ via the formula

$$h(\underline{y}, \underline{x}) = \frac{1}{2} \sum_{n=1}^{N} y_n^2 + V(\underline{x}) = \frac{1}{4N} \sum_{n,m=1}^{N} (p_n - p_m)^2 + V(\underline{q})$$

so that

$$H(\underline{p}, \underline{q}) = \frac{P^2}{2N} + h(\underline{y}, \underline{x}) .$$

Note that this definition of the relative-motion Hamiltonian $h(\underline{y}, \underline{x})$ entails that it Poisson commutes with both *P* and *Q* :

$$[P, h] = 0, \; [Q, h] = 0.$$

For completeness and future reference let us also display the equations of motion implied by the original Hamiltonian $H(\underline{p},\underline{q})$:

$$q_n' = p_n\,,\;\; p_n' = -\partial V(\underline{q})/\partial q_n\,,\;\; q_n'' = -\partial V(\underline{q})/\partial q_n\,,$$

where (for reasons that will be clear below) we denote as τ the independent variable corresponding to this Hamiltonian flow and with appended primes the differentiations with respect to this variable:

$$q_n \equiv q_n(\tau),\;\; p_n \equiv p_n(\tau),\;\; q_n' \equiv \partial q_n(\tau)/\partial\tau\,,\;\; p_n' \equiv \partial p_n(\tau)/\partial\tau\,.$$

Hence

$$Q' = \frac{P}{N}\,,\;\; P' = 0$$

yielding

$$Q(\tau) = Q(0) + \frac{P(0)}{N}\tau\,,\;\; P(\tau) = P(0),$$

as well as

$$x_n' = y_n = \frac{\partial h(\underline{y},\underline{x})}{\partial y_n}\,,\;\; y_n' = -\frac{\partial V(\underline{x})}{\partial x_n} = -\frac{\partial h(\underline{y},\underline{x})}{\partial x_n}\,.$$

Note that these equations have the standard Hamiltonian form even though, as mentioned above, x_n and y_n are not canonically conjugated variables.

This ends the review of quite standard results for the classical nonrelativistic many-body problem. Let us also emphasize that, above and below, the restriction to unit-mass particles, and to one-dimensional space, is merely for simplicity: generalizations – also of the following results – to the more general case with *different* masses and *arbitrary* space dimensions is quite elementary, essentially trivial.

The isochronous Hamiltonian

The Ω-modified *isochronous* Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$ is now defined by the formula

$$\tilde{H}(\underline{p},\underline{q};\Omega)=\frac{1}{2}\left\{\left[P+\frac{h(\underline{y},\underline{x})}{b}\right]^2+\Omega^2\,Q^2\right\},$$

where *b* is an arbitrary constant (introduced for dimensional reasons: it has the dimensions of a momentum, hence of the square-root of an energy) and Ω is a *positive* constant. Let us emphasize that hereafter the evolution of the various quantities is that caused by this new Ω-modified Hamiltonian; the corresponding independent variable is hereafter denoted as *t* (and interpreted as "time"), and differentiations with respect to this variable will be denoted, as usual, by superimposed dots, and of course, for any function $F\equiv F(\underline{p},\underline{q})$ of the canonical variable its time evolution will be determined by the standard equation

$$\dot{F}=\left[\tilde{H},F\right].$$

Solution of the isochronous Hamiltonian $\tilde{H}\left(\underline{p},\underline{q};\Omega\right)$

$$Q(t)=Q(0)\cos(\Omega\,t)+\dot{Q}(0)\frac{\sin(\Omega\,t)}{\Omega}=bC\,\frac{\sin[\Omega(t-t_0)]}{\Omega},$$

$$P(t)=P(0)\cos(\Omega t)+\dot{P}(0)\frac{\sin(\Omega t)}{\Omega}+\frac{h\left[\underline{y}(0),\underline{x}(0),\right]}{b}[\cos(\Omega t)-1],$$

$$\tilde{x}_n(t)=x_n(\tau),\ \ \tilde{y}_n(t)=y_n(\tau),$$

where (changing for convenience notation) we now denote as $\tilde{x}_n\,,\tilde{y}_n$ the canonical variables whose time evolution is determined by the Ω-modified Hamiltonian $\tilde{H}\left(\underline{p},\underline{q};\Omega\right)$ and as $x_n\,,y_n$ the canonical variables whose time evolution is determined by the original, un-modified Hamiltonian $H\left(\underline{p},\underline{q}\right)$. And here (most importantly)

$$\tau\equiv\tau(t)=\left\{A\sin(\Omega\,t)+B\left[1-\cos(\Omega\,t)\right]\right\}/\,\Omega\ ,$$

where the constants *A* and *B* are given by simple explicit formulas in terms of the initial position and velocity of the centre-of-mass of the system and of the Hamiltonian $\tilde{H}\left(\underline{p},\underline{q};\Omega\right)$ (which is of course a constant of motion). The crucial observation is that $\tau(t)$ (hence the entire solution) is a periodic function of *t* with period

$$T=2\pi\,/\,\Omega\ .$$

Behavior of the *isochronous* system over time scales much shorter than *T*

The solution formulas displayed above demonstrate that the dynamics yielded by the Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$ does not differ --- on a time scale short with respect to the period *T* --- from that yielded by the original Hamiltonian $H(\underline{p},\underline{q})$ (up to a *constant* rescaling of time). Indeed clearly $\tau=\tau(t)$ on a sufficiently short time scale varies linearly in *t,* since in the neighbourhood of any time $\bar{t}$ --- except when $\dot{\tau}(\bar{t})=C\cos(\Omega\bar{t})$ vanishes ---

$$\tau(t)=\overline{C}+t\,C\cos(\Omega\bar{t})+O\left[\left(\frac{t-\bar{t}}{T}\right)^2\right],$$

$$\overline{C}=C\left[\sin(\Omega\bar{t})-\Omega\bar{t}\cos(\Omega\bar{t})\right]/\Omega\,.$$

Transient chaos

One therefore finds that – essentially throughout the time evolution -- the Ω-modified dynamics differs from the unmodified one solely by a time rescaling -- by a possibly negative coefficient -- and by a time shift. The coefficient and the shift are time-independent over a time scale much smaller than the *isochrony* period $T=2\pi/\Omega$, but vary periodically with period T. A peculiar state of affairs arises, however, whenever $d\tau/dt$ changes its sign: this of course happens twice within every time period T, this being in fact a consequence of the periodicity of $\tau(t)$, which itself is the cause of the *isochrony*.

It is interesting to speculate on the application of this Ω-modification technique to any (translation-invariant) Hamiltonian describing a "realistic" translation-invariant many-body problem featuring, in its centre-of-mass system, "chaotic" motions with a natural time scale T_C. Then --- provided the constant Ω is assigned so that the *isochrony* period $T = 2\pi/\Omega$ is very much larger than this time scale, $T >> T_C$ --- the Ω-modified problem shall exhibit some kind of *chaotic* behavior for quite some time before the *isochronous* character of all its motions takes over, causing thereafter a recurrent evolution. This phenomenology --- qualitative rather than quantitative as described here, since a precise definition of *chaos* requires generally that a system displaying it be observed for *infinite* time --- is nevertheless remarkable.

The quantum case

Finally we tersely show that, in a quantal context, our Ω -modified Hamiltonian

$$\tilde{H}\left(\underline{p},\underline{q};\Omega\right)=\left\{\left[P+h\left(\underline{y},\underline{x}\right)/b\right]^2+\Omega^2 Q^2\right\}/2\,,$$

features an (infinitely degenerate) equispaced spectrum with spacing $\hbar\Omega$.

This spectrum consists of the eigenvalues E_k of the stationary Schrödinger equation

$$\frac{1}{2}\left\{\left[-i\hbar\frac{\partial}{\partial Z}+\frac{\lambda}{b}\right]^2+\Omega^2 Z^2\right\}\Psi_k\left(Z;\lambda\right)\psi_\lambda\left(\underline{z};\lambda\right)=E_k\Psi_k\left(Z;\lambda\right)\psi_\lambda\left(\underline{z};\lambda\right)\,,$$

obtained from this Hamiltonian via the standard quantization rule

$$P\Rightarrow -i\hbar\partial/\partial Z\,,\;\; Q\Rightarrow Z\;\;;\;\; \underline{p}\Rightarrow -i\hbar\partial/\partial\underline{z}\,,\;\; \underline{q}\Rightarrow\underline{z}\quad ,$$

and by identifying *λ* as an eigenvalue of the quantized version of the relative-motion Hamiltonian $h\left(\underline{y},\underline{x}\right)$. Indeed this Schrödinger equation is obtained by assuming that the eigenfunctions of the quantized version of the Hamiltonian $\tilde{H}\left(\underline{p},\underline{q};\Omega\right)$ factor into the product of an eigenfunction, $\Psi_k\left(Z;\lambda\right)$, depending on the variable *Z* and on which acts the differential operator *∂/∂Z* , and of the eigenfunction $\psi_\lambda\left(\underline{z};\lambda\right)$,corresponding to the eigenvalue *λ* of the quantized version of the relative-motion Hamiltonian $h\left(\underline{y},\underline{x}\right)$. The justification for this factorization is in the commutativity of the operators representing the quantal versions of the canonical variables *P* and *Q* , see above, with the operator representing the quantal version of the relative-motion Hamiltonian $h\left(\underline{y},\underline{x}\right)$ -- a commutativity reflecting the Poisson-commutativity of the corresponding quantities in the classical context.

It is now plain that the above Schrödinger equation features the spectrum and eigenfunctions

$$E_k=\hbar\Omega\left(k+1/2\right),\;\; k=0,1,2,\ldots,$$

$$\psi_k\left(Z;\lambda\right)=\exp\left(\frac{i\lambda z}{b\sqrt{\hbar\Omega}}-\frac{z^2}{2}\right)H_k\left(z\right)\,,\;\; z=Z\sqrt{\frac{\Omega}{\hbar}}\quad ,$$

where $H_k\left(z\right)$ denotes the standard Hermite polynomial of order *k*. This spectrum is of course equispaced with spacing $\hbar\Omega$, and it is infinitely degenerate inasmuch as it does not feature any dependence on the eigenvalues λ.

How to extend an *arbitrary* dynamical system so that the dynamics of the extended system is *isochronous* with period T , and moreover, over a fraction $(1-\varepsilon)\,T$ of that period (with ε arbitrary in the open interval $0<\varepsilon<1$), it is *exactly identical* to that of the original model

$$\underline{X}\,'=\underline{h}\left(\underline{X}\right);\;\; X_n\,'=h_n\left(\underline{X}\right),\;\; n=1,\ldots,N\,,\;\; (*)$$

$$\underline{\dot{x}}=\dot{\tau}\left(t\right)\,\underline{h}\left(\underline{x}\right);\;\; \dot{x}_n=\dot{\tau}\left(t\right)\,h_n\left(\underline{x}\right),\;\; n=1,\ldots,N\,,\;\; (**)$$

$$\underline{x}\left(t\right)=\underline{X}\left[\tau\left(t\right)\right];\;\; x_n\left(t\right)=X_n\left[\tau\left(t\right)\right],\;\; n=1,\ldots,N\,,$$

where $\underline{X}(\tau)$ is the general solution of (*). **Hence** $\tau\left(t+T\right)=\tau\left(t\right)$ **implies** $\underline{x}\left(t+T\right)=\underline{x}\left(t\right)$.

In conclusion, whenever $\underline{h}\left(\underline{X}\right)$ is such that the solution of the system (*) of *N* nonlinear ODEs exists globally, *all* solutions of the system (**) are periodic with period *T*: this system is ***isochronous***.

However, the system (**) is *not* autonomous. To eliminate this "defect", we perform the second step of our treatment, replacing this system (**) with the system

$$\dot{\underline{x}} = \phi\, \underline{h}(\underline{x});\ \ \dot{x}_n = \phi\, h_n(\underline{x}),\ \ n = 1,\ldots,N,\ \ (***)$$

which is of course equivalent provided the time-evolution of the scalar quantity ϕ is such that

$$\phi(t) = \dot{\tau}(t)\,,\ \ \tau(t) = \int_0^t dt'\,\phi(t')\,.$$

To construct a function $\phi(t) = \dot{\tau}(t)$ having the desired properties one may proceed as follows.

$$\dot{f}_1(t) = \Omega\, f_2(t),\ \ \dot{f}_2(t) = -\Omega\, f_1(t)\,;\ \ T = 2\pi/\Omega\,;$$

$$\phi(f_1, f_2; \varepsilon) = 1 - K(\varepsilon)\,\theta[S(f_1, f_2; \varepsilon)]\ \exp\left\{-[S(f_1, f_2; \varepsilon)]^{-2}\right\},$$

$$\theta(x) = 1\ if\ \ x > 0\,,\ \ \theta(x) = 0\ if\ \ x < 0\,,$$

$$K(\varepsilon) = \frac{1}{\varepsilon}\left(\int_0^1 dx\ \exp\left\{-\left[\sin^2\left(\frac{\varepsilon\pi}{2}\right) - \sin^2\left(\frac{\varepsilon\pi}{2}x\right)\right]^{-2}\right\}\right)^{-1},\ \ 0 < \varepsilon < 1\,,$$

$$S(f_1, f_2; \varepsilon) = -\frac{f_1^{\,2}}{f_1^{\,2} + f_2^{\,2}} + \sin^2\left(\frac{\varepsilon\pi}{2}\right)\ .$$

Solution:

$$\underline{x}(t) = \underline{X}[\tau(t)]; \;\; x_n(t) = X_n[\tau(t)], \;\; n = 1,\ldots, N \,,$$

$$f_1(t) = A \sin(\Omega t - \eta\pi) \,, \;\; f_2(t) = A \cos(\Omega t - \eta\pi) \,,$$

$$0 \le \eta < 1 \,; \quad f_{1,2}(t \pm T) = f_{1,2}(t) \,; \quad T = 2\pi / \Omega \,;$$

$$S[f_1(t), f_2(t); \varepsilon] = S(t) = -\sin^2(\Omega t - \eta\pi) + \sin^2(\varepsilon\pi / 2) \,,$$

$$S(t \pm T/2) = S(t), \;\; S(T_\pm) = 0, \;\; T_\pm = (\eta \pm \varepsilon/2) T/2 \,;$$

$$\phi(t) = 1 - \theta[S(t)]\, \dot{\Psi}(t) \,, \;\; \Psi(t) = K(\varepsilon) \int_0^t dt' \exp\left\{-[S(t')]^{-2}\right\} \,,$$

$$\phi(t \pm T/2) = \phi(t), \;\; \int_0^{T/2} dt\, \phi(t) = 0 \,; \;\; \tau(t \pm T/2) = \tau(t) \,.$$

If $\eta \geq \varepsilon/2$, hence $0 \leq T_- < T_+ < T/2$, then

$$\tau(t) = t \quad for \quad 0 \leq t < T_- \,,$$

$$\tau(t) = t - \Psi(t) + \Psi(T_-) \quad for \quad T_- \leq t < T_+ \,,$$

$$\tau(t) = t - T/2 \quad for \quad T_+ \leq t < T/2 \,;$$

if instead $\eta \leq \varepsilon/2$, hence $T_- \leq 0 < T_+ < T/2$, then

$$\tau(t) = t - \Psi(t) + A \quad for \quad 0 \leq t < T_+ \,,$$

$$\tau(t) = t + A - B \quad for \quad T_+ \leq t < T_- + T/2 \,,$$

$$\tau(t) = t - \Psi(t) + A - B + C \quad for \quad T_{-.} + T/2 \leq t < T/2 \,;$$

$$A = \Psi(0),\ B = \Psi(T_+)_-,\ C = \Psi(T_- + T/2).$$

Note that in both cases $\tau(t)$, besides being *periodic* with period *T/2*, is *linear* in *t* over a fraction $1-\varepsilon$ of its period *T/2*.

Outlook

We have seen that, given an arbitrary (autonomous) dynamical system --- possibly characterized by a *chaotic* dynamics (according to the standard definitions of *chaotic* behavior) --- it is possible to extend it so as to obtain thereby a new (autonomous) dynamical system which is *isochronous* (all its solutions are completely periodic with an *a priori* fixed period) hence it is *integrable* (indeed, “more than superintegrable”: according to the standard definition of superintegrable systems, as possessing the maximal number of functionally independent constants of motion, so that *all* their confined solutions are *completely periodic*, but not necessarily *isochronous*); yet it is also such that, over an *arbitrarily large* fraction (of course, less than unity) of its period it reproduces *exactly* the dynamics of the original model. This fact suggests the need to invent new definitions of *chaotic* behavior (or perhaps, using a new name, of “*complex*” behavior) which do not refer to the behavior of a system over *infinite* time (as the current definition of *chaotic* behavior does), but rather equip this new definition with an associate *time scale* characterizing its validity. Note that, to some extent, this is now being done in the *integrable* case via the introduction of multiple scale analysis and the recognition that a *nonintegrable* system may nevertheless be *more or less nonintegrable* depending on the order of a multiple scale reduction of it that yields an *integrable* dynamics (indeed *all* dynamical systems are *integrable* over a sufficiently short time scale).